Global Perturbative QCD Analysis of Fragmentation Functions and Parton Helicity Distributions

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Quantum Chromodynamics (QCD) is a part of the standard model of the elementary particles. QCD is an unbroken non-Abelian gauge theory of the strong interaction based on local gauge symmetry of inner $SU(3)$ color freedom. Fundamental ingredients of QCD are quarks and gluons, generally called partons. A quark is a spin $1/2$ fermion carrying color triplet index, and a gluon is a spin $1$ vector gauge boson carrying color octet index. Because of its non-Abelian nature, QCD has a peculiar feature of asymptotic freedom. This enables us to calculate high energy interactions between quarks and gluons within the perturbative treatment. This is called perturbative QCD. On the other hand, in the low energy region of less than 1 GeV, QCD shows color confinement. This confines all the color freedoms of quarks and gluons into the spectrum of color singlet hadrons in physically observable states. The process of the hadronization cannot be treated perturbatively.

However, there is a remedy in the effective description of high energy interactions including hadrons. The color confinement is difficult to treat, but it can be effectively avoided by introducing several parton distribution functions. Low energy behavior, or equivalently long distance behavior, of quarks and gluons can be absorbed into the definition of these distributions. The extraction of the long distance behavior can be performed systematically. Most importantly the long distance part can be factored out from the rest of the parton interaction which describes high energy, or short distance, parton interaction. The distributions contain all the long distance information on the parton structure of hadrons. This factorial separation between the short and long distance behavior is called factorization. The factorization property is proved in many high energy processes based on the general property of infrared behavior of quarks and gluons in perturbative QCD. By virtue of the generality, the parton distributions has process independent nature. This process independence is called universality of the parton distributions.

Because of the generality of the factorization and the universality, several high energy interactions in wide range of kinematics can be analyzed in a general framework of perturbative QCD. This is called global perturbative QCD analysis. We can extract the parton distributions from various experimental data and study the inner parton structure of hadron through the distributions.
First I tried to understand firmly the general property of perturbative QCD and its applicability to various high energy processes. Then, I constructed the new numerical calculation framework based on the perturbative QCD with the Mellin transform technique. The Mellin transformation is equivalent to the Laplace or Fourier transformation. I implemented an effective calculation method on the foundation of the general property of the Laplace transform. With this framework, I could achieve several $\chi^2$ fits to experimental data including various complex processes within reasonable computation time. It became at most thousand times shorter than the time needed in my previous framework in which Mellin transformation was not used. Not only the created values but also the uncertainties of the fitted parameters are investigated. In our framework, statistical errors coming from experimental errors are treated properly by Hessian method or Lagrange multiplier method.

I applied the general framework to the analyses of the experimental data; 1) inclusive hadron production in high energy $e^+ e^-$ collision, 2) semi-inclusive deep inelastic lepton scattering, 3) deep inelastic lepton scattering on longitudinally polarized target, and 4) semi-inclusive deep inelastic lepton scattering on longitudinally polarized target. From the analysis of the first two processes, I could effectively extract the parton fragmentation functions. These provide us the information on the number of hadrons created from a parton in the hadronization process. From the analysis of the last two processes, I could effectively determine parton helicity distributions with the help of the fragmentation functions extracted beforehand. The helicity distributions yield the information on the helicity contribution of a parton to the spin of a hadron, especially a proton. Through this analysis, I could investigate the helicity contribution of each parton. This provides a clear picture of the proton spin structure. This will help up to solve the problem of the spin puzzle which was firstly indicated by EMC experiment. The result will supply a standard for further studies on the angular momentum contributions of partons through generalized parton distributions. On the footing of detailed understanding of theoretical background, I studied the theoretical systematic errors on the final results.
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Chapter 1

Introduction

Quantum Chromodynamics (QCD) \[1\] is a Yang-Mills type quantum field theory \[4\] of the strong interaction with non-Abelian gauge fields. It mediates the local interactions between spin 1/2 fermion fields called quarks. The bosonic gauge vector fields with spin 1 are named gluons. The Lagrangian of QCD is simply derived by imposing the comprehensive principle of the renormalizability for the effective local field theory \[5\] and the local gauge invariance \[11\] on the inner SU(3) freedom, called color, of the quarks. Then the Lagrangian of QCD for the quark field $\psi$ which obeys the fundamental representation of SU(3) symmetry is written as

\[ L_{QCD} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \bar{\psi} (i \gamma^\mu D_\mu - m) \psi \]  
\[ D_\mu = (\partial_\mu - ig_s A_\mu^a T^a) \quad (T^a = \lambda^a/2) \]  
\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c, \]

where $\lambda^a$ is SU(3) Gell-Mann matrix, $m$ is the mass of the quark, $g_s$ is the strong gauge coupling, $a$ is color octet index and $f^{abc}$ is the structure constant of SU(3) group and the gauge field $A_\mu^a$ expresses the gluon which has octet color and transforms as an adjoint representation of SU(3). Here the gauge-fixing term and ghost term \[9\] \[12\] are omitted.

It is found that there are many quarks which are different only for their masses and all of these quarks are described equally in the framework of QCD. The number of quarks, called flavor, is now 6 (up, down, charm, strange, top, bottom) \[13\] \[14\] covering the three generations. (Each particle has of course its anti-particle \[15\].) The origin of the flavor symmetry breaking through the masses is thought to be in the domain of the electro-weak interactions \[16\] \[17\] \[18\] \[19\]. Thus the question of the origin of the flavor breaking is not adequate
to ask of a strong interaction theory. On the same footing of participation of quarks into the electro-weak interaction, there is the difference between quark flavors in the electro-weak sector. The up, charm, bottom quarks have the fractional electric charge of $2/3 \, e$, while the down, strange, bottom have $-1/3 \, e$, where $e$ is the magnitude of the charge of the electron. Along with the electro-weak gauge field theory, QCD composes the standard model of the elementary particles which was constructed based on the local gauge symmetry of $U(1) \times SU(2) \times SU(3)$.

More than 30 years have past since QCD appeared as a candidate for the strong interaction describing hadrons. Now it is strongly believed that hadrons are composed of the quarks and gluons, and QCD describes the fundamental dynamics of those constituents. It is also believed that QCD has the peculiar feature of color confinement, which says that asymptotic physical state in QCD dynamics observed as a particle must be the color singlet bound state of quarks and gluons. (It is easy to convince ourselves that the color singlet state means that the only physical states are integrally charged as observed hadron spectrum indicates $^{20}_{13}$.) Thus, the color degrees of freedom, or equivalently quarks and gluons as the fundamental ingredients of QCD, are in principle never observed directly and are confined in the hadron as the realization of the color singlet bound state in physically observed states. It can be said that the investigation of QCD is never separated from that of the inner structure of hadrons.

In the framework of the interacting field theory, the reliable well-known tool for its analysis is the perturbative method. As the name of strong interaction indicates, the coupling constant of QCD is large in the mass scale of hadrons ($\ll 1 \, \text{GeV}$) so the perturbative method cannot be applied to QCD in such energy scale. In spite of the important works on the color confinement, e.g., $^{21}_{21}$, $^{22}_{22}$, $^{23}_{23}$ the theoretical proof of the confinement in QCD from its first principle seems still an outstanding problem while the numerical calculation based on the lattice QCD $^{24}_{24}$, $^{25}_{25}$ strongly suggests its existence.

On the other hand, QCD as a non-Abelian field theory has another remarkable feature of asymptotic freedom $^{26}_{26}$, $^{27}_{27}$, which says that as the energy (momentum) scale of the system increases, the system described by QCD behaves almost as the free field theory. Soon after, it was revealed that the asymptotic free nature is only of the non-Abelian theory in four dimensions by exhausting possible renormalizable interactions in $^{28}_{28}$. The interactions between quarks are becoming weaker. Thus in the energy scale,
the coupling between quarks and gluons becomes weak enough (but still stronger than electromagnetic coupling) so that the behavior of the system is analyzable by applying the perturbative method to QCD on the same analytical footing with QED.

The asymptotic free feature of QCD dynamics or the scaling property as its results in high energy interactions played an critical role to obtain the common recognition that QCD is surely a theory of the strong interaction, which also takes the central role of this analysis. Other examples of successful contact between QCD and experimental observations in the incunabula can be found in \[29\]. The scaling property was first observed in the historical experiment of the deep inelastic scattering (DIS) in SLAC \[30\] \[31\] \[32\]. DIS is the typical high energy process where the an electron \(e\) is hardly scattered off a target hadron \(H\) to \(e'\) mediated by hard virtual photon. The hard scattering breaks the target and creates a system with a large number of hadrons \(X\), fig. \[1.1\].

\[
e + H \rightarrow e' + X .
\] (1.4)

The dominance of the DIS hard scattering region in phase space and the

![Figure 1.1: description of DIS in the parton model](image)

independence of its rate on the energy scale of the interaction indicate a hadron structure as an assemblage of loosely bound point-like constituents
in the high energy region $[33]$. Thus the observed scaling property and considerable scattering rate in the DIS experiment provided the first intuitive evidence of not only the requirement of the asymptotic free nature for the strong interaction but also the point-like structure inside of the hadron. The typical energy scale of the hard photon as a probe is $> 1$ GeV $\sim$ the proton mass scale. At this energy, the inner parton structure comes to be visible and parton level interactions dominates. The scaling property in DIS process was named Bjorken scaling $[33]$ reflecting his first prediction, and the point-like constituents of a hadron were named partons and were recognized as quarks and gluons along with the rigid confirmation of QCD as a field theory describing the strong interaction.

The loosely bound point-like parton picture for the hadron inner structure in high energy interactions was named the parton model $[33, 34]$. In the parton model, the deep inelastic scattering (DIS) process is described as fig. 1.1. In the DIS energy region, the interactions between partons in a target hadron, the blue circle in fig. 1.1 can be neglected and the target serves only as the source of a parton which interacts hardly with the virtual photon, the red circle in fig. 1.1. The function of the hadron as a parton source is described by a distribution function $f(x)$ which provides the probability density to find a $f$ type parton in the hadron with the (linear) momentum fraction $x$ of the parton against that of the original hadron. The probability density is called parton distribution function. Then the cross section of DIS process $\sigma(P)$ is schematically expressed as

$$\sigma(P) = \int_0^1 dx \sum_f f(x) \sigma(xP), \quad \text{factorization} \quad (1.5)$$

where $\sigma(xP)$ is the parton level cross section. The actual distribution of $f(x)$ depends on the concrete hadron composition which still seems to be difficult to predict from the first principle. (There are several studies on the part of the information on the distributions through the lattice QCD or other effective models, e.g., $[35, 36, 37, 38, 39]$.) The simple prescription of the parton model, incomplete though it is, for high energy interactions like DIS process illustrates formally well the observed experimental results.

The important fact is that the distribution which describes hadron composition and the parton level interaction which illustrates the hard interaction are effectively separated. They are connected by the integration on the probability argument $x$. Note that the hard interaction reflects the parton level short distance interaction while the hadron composition is determined in the
color confinement scale $\ll 1$ GeV thus the distributions are considered to contain the information on the long distance behavior of parton interaction. The effective separation of the short distance behavior from the long distance behavior is called the factorization, eq. (1.5). The important fact is that by virtue of the asymptotic freedom, we can calculate the short distance parton interactions by the perturbative method in QCD.

The introduction of the factorization, equivalently the parton model in DIS process seems to be intuitive and an ad hoc prescription in this stage. As I’m going to describe in the next chapter, we can however construct that factorization property on more field theoretical foundation. The basic idea is that the elimination of long distance behavior from the short distance parton interaction. (The long distance factor leads to the famous DGLAP scale evolution of the distributions [40].) As we can see in the next chapter, the long distance property of the interaction can be factored out, and absorbed into the definition of parton distribution functions. This proof of the factorization can be achieved systematically based on general Feynman diagrammatic treatment, and, for many processes, the factorization has been proven to all orders of the perturbative expansion of corresponding short distance interactions.

The crucial recognition is that the hadron composition (hadronization process) of partons and the long distance factor, which is actually eliminated as infrared divergences in the short distance interaction, is process independent. Thus the distributions should have the universal nature, sometimes called universality of the parton distributions. Because we can calculate the short distance interaction perturbatively in QCD, we can extract those distributions by contraries from many high energy processes where the parton model picture becomes significant in the framework of perturbatively treated QCD. Through the extracted distributions we can access the information on the inner parton structure of hadrons. As an index of hadrons, we usually focus on the proton structure. Because of the universality and the generality of the framework based on the general property of QCD dynamics, this study is called global perturbative QCD analysis.

Through the intensive studies, e.g., [41, 42, 43], on the momentum probability density distributions $f(x)$, called unpolarized parton distribution functions, now we have consistent reliable knowledge on the momentum structure of the proton mainly based on high precision data from HERA accelerator [44, 45]. Those distributions are applied as an input information for the analyses or predictions for the coming LHC physics [46, 47], for example.
We can also define several distributions describing some other information on the hadron (proton) structure. The distributions I’m going to focusing on in this analysis are parton helicity distributions, or simply (longitudinally) polarized parton distribution functions, which carries the information on the spin $1/2$ composition of the proton. The distributions will be be defined on firm basis in the next chapter.

The origin of the question on the spin structure of the proton appeared from the results provided in 1988 from EMC experiment [48] of longitudinally polarized deep inelastic scattering. The longitudinally polarized DIS is the DIS with longitudinally polarized target and probe lepton. Through the process, we can access the parton helicity distributions, or equivalently the contribution of the helicity of parton to the proton spin as I will explain in chapter 3. In simple quark model, the nucleon spin composition is expressed in short hand (with its permutations neglected) as follows by the exclusion principle on the quark fermion.

$$|p \uparrow\rangle = |u \uparrow u \uparrow d \downarrow\rangle, \quad |n \uparrow\rangle = |u \downarrow d \uparrow d \downarrow\rangle.$$  \hspace{1cm} (1.6)

Thus it is expected that the spin $1/2$ of the proton is carried roughly $+2/3$ part by up quark and $-1/6$ part by down quark, i.e., 133% by up quark and $-33\%$ by down quark and of course 100% in quark sector. However, according to the results and analyses by EMC experiment on the helicity distributions, it turned out that the contribution of quark sector had roughly only 20% contribution to the proton spin, in contrast to the quark model expectation. The smallness of the quark helicity contribution is called proton spin puzzle. Since that era, the spin structure of the proton has been extensively studied until now experimentally [49, 50, 51, 52, 53] and theoretically [37, 54, 55, 36].

However, unlike unpolarized parton distributions, the kinematic coverage of (longitudinally) polarized experiments are not wide thus far that we can pin down the helicity distributions only by the results from a few experiments. (Because handling the data from many experiments introduces several complexities in its analysis, the number of experiments is better to be limited ideally for more detail study.) Thus we need to introduce many different processes to extract them. Now in parton helicity distribution markets, comprehensive analyses including many kinds of intricate processes just started [56] to extract effectively the distributions, fig. 1.2. Some processes require us of the information on the fragmentation process of parton, which can be described by other distribution functions called fragmentation functions fig. 1.2 (The definition of the fragmentation functions is also given
in the next chapter.) For the comprehensive analyses for the helicity distributions, inclusion of the processes in the crossing region between parton distributions and fragmentation functions plays significant role. In our analysis, we also extracted the fragmentation functions to investigate the effect of the distribution onto the determination of helicity distributions. Then after we tried the effective extraction of parton helicity distributions with the input of the determined fragmentation functions. The analysis of the helicity distributions over the many processes therefore would be not only of the physical importance of those distributions, but also truly challenging issue by itself in sense of the application of perturbative QCD framework in high energy interaction. Moreover, better understanding of the contributions of parton spin would also have great significance for the coming studies on the contributions of angular momentum of partons to the proton spin, which will be referred in chapter 3.

![Perturbative QCD Analysis Diagram]

**Figure 1.2:** comprehensive treatment of various high energy processes in perturbative QCD

Based on firm understanding of theoretical background of perturbative QCD, I constructed a comprehensive analysis framework handling various processes and parton distributions. Table 5.1 shows the applicability of my
perturbative QCD framework to the analysis of various parton distributions. In my framework, unpolarized PDFs are used as an input. It is indeed possible to perform the fit of unpolarized PDFs in my framework, but it has been already determined well by other groups. Thus I did not include the possibility of this analysis in my framework in the table. As mentioned above, unpolarized PDFs are already well determined well almost only with the data of unpolarized deep inelastic scattering. On the other hand, the better analysis of longitudinally polarized PDFs, parton helicity distributions, requires the input of various processes. In my analysis framework, those processes can be comprehensively handled in consistent manner. In the analysis of this thesis, I concentrated on the detailed study of helicity distributions with the data related to polarized deep inelastic scattering, motivated by simultaneous treatment of fragmentation functions in the same framework. The inclusion of polarized proton-proton collision data is one of the prior extensions of my framework, and some works are now proceeding.

This thesis has the following structure. First I will begin with the introduction of the theoretical framework of the perturbative QCD analysis. I took considerable space for the general fundamental background because the deep understanding of it really had substantial significance for performing this analysis. The formal definition of the parton distributions will be given also there.

In chapter 3, I will deduce several results provided from the general perturbative QCD framework to make its application to my analysis clear. I will start from the DGLAP equations and introduce several processes treated in my analysis. The typical sum rules imposed on the various distributions are introduced there.

In chapter 4, I will introduce analysis method I applied. For this analysis I developed a new calculation framework based on the perturbative QCD. My primitive analyses \[57, 58\] of parton helicity distributions and fragmentation functions were performed in a calculation framework based on more straightforward calculation procedure. The framework encountered limitation on performing the required more comprehensive analysis. In the chapter, I will explain the basic idea of the new calculation framework first. Then I will take some space to introduce the \(\chi^2\) minimization fit as a fundamental tool to extract the distributions from experimental data. Then after, I will show the fundamental set up for the analysis including several conditions I set for the distributions and experimental data I included in this analysis.

In chapter 5, I will also show the results of my analyses separately for the fragmentation functions and the parton helicity distributions. The prospects with those extracted distributions to future experiments will be also given.
In chapter 6, I will summarize the results and provide the conclusions of this analysis.

In appendices, I will develop the general definition of spin in relativistic field theory and introduce several interesting issues related to spin, especially spin-statistics theorem. The understanding of these fundamental nature of spin gave firm ground for this analysis.

♦ NOTATIONS

In this thesis, I’m going to use the following notations, generally based on Bjorken and Drell [59].

- Natural units $\hbar = c = 1$ are used throughout.
- The basic unit of charge is the magnitude of the charge of the electron: $e > 0$ and $Q_q$ is a charge of a particle $q$ expressing $Q_q = e_q e$ with $e_q$ fractional charge value of the particle.
- Greek indices, $\mu, \nu, \ldots$, run from 0 to 3 and Latin indices, $i, j, \ldots$, run from 1 to 3.
- Repeated indices are summed in all cases.
- The Minkowski metric $g^{\mu\nu}$ is $\{1, -1, -1, -1\}$ in diagonal elements.
- The totally antisymmetric (Levi-Civita) tensor $\epsilon^{\mu\nu\rho\sigma}$ is defined so that $\epsilon^{0123} = +1$.
- The scalar product of the $\gamma$ matrix and any 4-vector $A$ is expressed as $\bar{A} = \gamma_\mu A^\mu$.
- Four vectors are denoted by light italic type and three vectors by boldface type ($x^\mu = (x^0, \mathbf{x})$).
- For an (asymptotic) one-particle state $i$ in momentum space, assume $p_i^\mu = (p_i^0 = E_i > 0, \mathbf{p}_i)$ with $p^2 = m_i^2 \geq 0$.
- $\alpha_i$ which appears in perturbative expansion is recognized as $g_i^2/4\pi$ with $g_i$ a coupling constant of a theory.
- Let $N_c$, $C_F$, $C_A$, $T_R$ (and $T_F$) parameters originating from the color $SU(3)$ group. There are given respectively as $N_c = C_F = 3$, $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$, and $T_F = T_R n_f = \frac{1}{2} n_f$ with $n_f$ the number of active flavors.
Table 1.1: Table of applicability of my perturbative QCD framework to the analysis of various parton distributions compared with other representative analyses classified by applied experimental data; △ denotes possible future extension.

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<th>DIS polarized</th>
<th>$e^+e^-$ collision FF</th>
<th>semi-inclusive DIS unpolarized</th>
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<th>p-p collision</th>
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Chapter 2

Theoretical Background of Perturbative QCD

In this chapter, I’m going to develop the formal theoretical framework of perturbative QCD (pQCD) to be applied to study parton structure of hadrons. The key observation that makes us apply pQCD description to high energy collisions is that high energy interactions of partons, which is described by perturbation method by virtue of asymptotic freedom, can be effectively factorized from short distance information on the parton structure of hadrons, described as distribution functions of the hadrons, as it to be the parton model. eq. \[ \text{eq. 1.5} \] The explanation of this factorization nature on more theoretical grounds is the main issue of this chapter. We will see this nature appears naturally in the process of the QCD radiative corrections to the high energy interactions of partons, where the dominant contribution is consistent with the parton model description in no correction limit. Therefore such a description with the factorization property predicts qualitative simplifications of the behavior of the partons at high energy and also a specific pattern of corrections to this behavior.

In the first section, I’m going to review a primitive method of theoretical illustration of the factorization, which was successfully implemented for the calculation of radiative corrections to deep inelastic scattering (DIS) process with the help of operator product expansion and renormalization group equation. In the successive sections, I’m going to introduce in some detail an alternative way of theoretical description, called mass factorization, which has great advantage to be able to applied to the account of much general processes because of its more general diagrammatic (analytical) nature. To make the difference of the two schemes clear, though, I’m going to concentrate on the discussion on DIS process even in those sections. We will see that
the two methods gives the same physical picture for DIS. Its generalization to other processes would be obvious.

2.1 Appearance of Factorization and Radiative Corrections

Historically, the theoretical description of the factorization property was firstly resolved with Wilson’s operator product expansion (OPE) \[70\] for the calculations of radiative corrections to DIS process. By using the property of OPE, we can arrive at the results consistent with the the parton prescription.

DIS is a high energy process, where the initial electron $e$ with momentum $k\mu$ scattered off the target hadron $H$ with momentum $P\mu$ to $e'\gamma$ with momentum $k'\mu$. The hard scattering, defined as $-q^2 = Q^2 = -(k - k')^2 \gg P^2$, breaks the target and creates a system with a large number of hadrons $X$, fig. 2.1

$$e + H \rightarrow e' + X \tag{2.1}$$

The cross section of DIS process is in general expressed without going into the parton picture as

$$\sigma(ep \rightarrow e'X) = \frac{2\pi e^4}{s} \int \frac{d^3 k'}{(2\pi)^3 2k_0'} \frac{1}{q^2} L_{\mu\nu}(k, k') W^{\mu\nu}(P, q) \tag{2.2}$$

$$L_{\mu\nu}(k, k') = \frac{1}{2} \sum_{\text{spins}} \{[\bar{u}(k)\gamma_{\mu}u(k')]\bar{u}(k')\gamma_{\nu}u(k)\} \tag{2.3}$$

$$W^{\mu\nu}(P, q) = \frac{1}{4\pi 2} \sum_{X} \int d\Pi_{X} \langle P|J^{\mu}(-q)|X\rangle \langle X|J^{\nu}(q)|P\rangle , \tag{2.4}$$

where $X$ is the possible final hadron systems and $J^{\mu}$ is the electromagnetic current of quarks, $J^{\mu} = \sum_{f} Q_{f}\bar{q}_{f}\gamma^{\mu}q_{f}$, where the sum runs over the quark flavors $f$ and $Q_{f}$ is their electric charge. Then, with the help of the optical theorem, fig. 2.2 we deduce to the evaluation of the matrix element of the product of operators $J^{\mu}$.

$$\int d^{4}xe^{iq\cdot x}\langle P|T(J^{\mu}(x)J^{\nu}(0))|P\rangle . \tag{2.5}$$

In DIS $q$ (exactly $q^2$) becomes so large that the region $x\rightarrow 0$ becomes more dominant in the integral. A product of operators located at almost the same space-time point is generally known to be decomposed into the sum of composite (local) operators. This expansion to the sum of local operators is called operator product expansion (OPE), which was proposed in \[70\].
2.1.1 Operator Product Expansion (OPE)

The operator product expansion (OPE) is expressed as

\[ A(\xi/2)B(-\xi/2) \overset{\xi \to 0}{\sim} \sum_i C_i(\xi)O_i(0), \tag{2.6} \]

where \( A, B, O \) are composite operators and \( \xi \) is an arbitrary space-time vector. The most important property of OPE is that the coefficient \( C_i(\xi) \) behaves as

\[ C_i(\xi) \overset{\xi \to 0}{\sim} \left( \frac{1}{\xi} \right)^{d_A+d_B-d_i}, \tag{2.7} \]

where \( d_A, d_B, d_i \) are the dimensions of the composite operators \( A, B, O \) respectively. Therefore in the short distance limit \( \xi \approx 0 \), the behavior of the operator product is dominated by the first few lower dimension terms in the OPE. The proof of OPE was first made in [71] based on a renormalization method, called BPHZ method now. It is also well summarized in [72].

In case of DIS, correctly speaking, the product of the current operator must be considered in the limit \( \xi^2 \to 0 \). This limit is called “light-cone limit”.
Then OPE receives a slight modification as

$$A(\xi^2) B(-\xi^2) \approx \sum_{i,n} C_n^i(\xi^2) \xi^{\mu_1} \cdots \xi^{\mu_n} O^{i}_{\mu_1 \cdots \mu_n}(0). \quad (2.8)$$

Then the behavior of the coefficient $C_i^n(x)$ becomes

$$C_i^n(\xi^2) \xi^{\mu_1} \cdots \xi^{\mu_n} \sim \left(\frac{1}{\xi^2}\right)^{(d_A+d_B-d_i+n)/2}. \quad (2.9)$$

determining the dominant terms is called “twist”. The higher twist term becomes less dominant.

For the case of DIS, the lowest twist (twist 2) operators which appear in the light-cone OPE of the product of the quark current $J^\mu$, eq. (2.5) are given gauge invariant way as

$$O^F_{\mu_1 \cdots \mu_n} = i^{n-1} \bar{q} S(\gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_n}) q - \text{traces}, \quad (2.10)$$
$$O^V_{\mu_1 \cdots \mu_n} = i^{n-2} S(\lambda^a_{\mu_1} D_{\mu_2}^{a_1 a_2} \cdots D_{\mu_n-2}^{a_2 a_3} F_{\mu_n}^{a_{n-1} \nu}) - \text{traces} \quad (2.11)$$

where $D_{\mu}$ is the covariant derivative eq. (1.2) and the bracket with the symbol “$S$” means making the covariant indices $\mu_1 \cdots \mu_n$ symmetric and “traces” means subtracting terms proportional to $g_{\mu \nu}$ to make the operator traceless on any pairs of the covariant indices. Because of the dominant contribution of these operators, these lowest twist operators can be identified with the parton model description (though we still need dispersion relations to obtain the expression for DIS kinematics region.) In fact, if we keep the intuitive parton model description as much as possible, the operators can be recognized as sum of the moments of parton distributions, as briefly reviewed in [73]. We also see this identification more apparently in the proceeding sections of mass factorization.

Thus now we can see the factorization of parton picture in this framework. As the space-time point of $A$ approaches to that of $B$, the The behavior of the product is expressed in terms of $C_i$, reducing its contribution by the dimension of the corresponding $O_i$. Therefore it can be said that $C$ reflects the short distance behavior of the operator product, contrast to $O$ which reflects the long distance behavior which is absorbed into the parton distribution functions. Consequently it is concluded that OPE gives an appropriate tool to factor the effective short distance behavior, which dominates the behavior
of the operator product in the short distance limit, from the long distance behavior. This justifies the naive parton model picture, eq. 1.5.

Correctly, OPE must be treated between renormalized quantities because the local operator like A, B, O is not well-defined and includes divergence in the field theory as OPE implicitly indicates. These divergences must be renormalized with newly introduced appropriate renormalization constants. As the result of the radiative corrections, the naive dimension counting, like eq. 2.9 receives logarithmic corrections of the type \( \log \left( \frac{Q^2}{\mu^2} \right) \), where \( Q^2 \) is the Fourier inverse of \( \xi^2 \) and \( \mu \) is the renormalization scale. To see this behavior clearly, it is convenient to use renormalization group equation (RGE). Application of RGE has another significance of summing up \( (\alpha_s \log \left( \frac{Q^2}{\Lambda^2_{QCD}} \right))^n \) corrections, which appears in \( n \)th loop calculations. Because \( \alpha_s \log \left( \frac{Q^2}{\Lambda^2_{QCD}} \right) \) is the order of 1 in QCD case ( \( \Lambda_{QCD} \sim \) a few hundred MeV ), the summation is needed to make any quantitative prediction with the radiative corrections.

### 2.1.2 General Property of Renormalization Group Equation (RGE)

Before going to the application of the renormalization group equation (RGE) to the OPE results, let me introduce the conventional discussion of the general property of RGE to make the meaning of \( \lambda_{QCD} \) and the renormalization scale \( \mu^{(0)} \) clear. Renormalization procedure in a field theory consists of regularization of divergences, which appear as quantum loop corrections, and redefinition of ingredients of the theory so as to make the observables finite. For the moment, let me concentrate on \( \phi^4 \) theory. If we consider \( \Gamma \) as the generating functional of the truncated one-particle-irreducible (1PI) Green’s functions \( \Gamma^{(n)} \), it can be expressed as

\[
\Gamma \left[ \phi; m^2, \lambda; \mu^2 \right] = \Gamma_0 \left[ \phi_0; m_0^2, \lambda_0; \Lambda \right]
\]

\[
\phi_0 = Z_{\phi}^{1/2} \phi, \quad m_0^2 = Z_m m^2, \quad \lambda_0 = Z_\lambda \lambda,
\]

\[
Z_i = Z_i(\lambda_0, m_0/\Lambda, \Lambda/\mu) \quad (i = 3, m, \lambda),
\]

where quantities with subscript 0 means the bare quantities appearing original Lagrangian and are related to its renormalized quantities with (dimensionless) renormalization constants \( Z, \Lambda \) is a some regulator of the divergence and \( \mu \) is the renormalization scale expressing the mass scale where the renormalization constants are defined by renormalization conditions. Renormalizability simply says that if bare ingredients \( (\phi_0, m_0, \lambda_0) \) are redefined as eq. 2.13 with the renormalization constants eq. 2.14, the divergent quantity
(Γ₀ in the right hand side of eq. 2.12) can be rewritten as the finite function of the renormalized ingredients (Γ in the left side of eq. 2.12). To regularize divergences, dimensional regularization method [74] is commonly used because it keeps gauge invariance in the process of renormalization. In the dimensional regularization, the regulator is given by the dimension ε = 4 − d instead of Λ. Then the renormalization constants Zᵢ depends only on ε and λ₀, i.e. Zᵢ(λ₀, ε). To fix the renormalization constants, renormalization condition based on mass independent renormalization (MIR) like MP scheme [75] is applied because of the ease of treatment. MIR treats masses of a theory as some input parameters so as coupling constants. This makes the handling of RGE much simpler and is appropriate for QCD which has no on-shell asymptotic definitions of quarks.

Now that renormalization the renormalized quantity Γ does not have divergence any more, but it still has a freedom to choose the value of µ reflecting the way of renormalization or redefinition. A quantity defined with a scale µ is different from that with a different scale µ′. The important fact is that, if renormalization argument starts from one original theory, a physical (renormalized) quantity, like Γ, must be unique and should never depends on the choice of µ. This implicitly indicates it would be possible to rewrite the physical quantity given with µ to that with µ′ by a (finite) transformation between ingredients, like φ⁰(µ₀) ↔ φ(µ).

\[
\Gamma' [\phi'; m^2, \lambda'; \mu^2] = \Gamma [\phi; m^2, \lambda; \mu^2] = \Gamma_0 [\phi_0; m_0^2, \lambda_0; \Lambda].
\]  (2.15)

finite transformations between ingredients

It turns out that the set of these transformations has the group property. This group property of the renormalized quantities defined with the different scales, or more generally renormalization schemes, is called renormalization group. When the argument is restricted to a continuous group composed of the renormalized quantities with the different scale µ, these quantities must obey the following differential equation expressing the response of the quantity to the infinitesimal change of the scale µ. This differential equation is called renormalization group equation (RGE).

The renormalization group equation is derived just from the fact that the right hand side of eq. 2.12 does not depend on the renormalization scale µ.

\[
\left( \mu \frac{\partial}{\partial \mu} \right)_0 \Gamma_0 [\phi_0; m_0^2, \lambda_0; \Lambda] = 0.
\]  (2.16)
Then the following differential equation is derived from the left side of eq. (2.12)

$$
(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma_m(\lambda) m^2 \frac{\partial}{\partial m^2} - \gamma(\lambda) \frac{\delta}{\delta \phi} ) \Gamma[\phi; m^2, \lambda; \mu^2] = 0 , \quad (2.17)
$$

where $\beta$, $\gamma_m$ and $\gamma$ are finite functions reflecting the finite transformations and defined as

$$
\beta(\lambda) \equiv \left( \mu \frac{\partial}{\partial \mu} \right)_0 \lambda \quad (2.18) \\
\gamma_m(\lambda) \equiv - \left( \mu \frac{\partial}{\partial \mu} \right)_0 \ln m^2 \quad (2.19) \\
\gamma(\lambda) \equiv - \left( \mu \frac{\partial}{\partial \mu} \right)_0 \ln \phi , \quad (2.20)
$$

where the differentiation $(\mu \partial/\partial \mu)_0$ means the derivation with $\phi, m_0, \lambda_0, \Lambda$ fixed (the original theory fixed) and $\phi \delta/\delta \phi$ is the functional derivative of $\phi$. $\beta, \gamma_m, \gamma$, called the renormalization group functions, are independent of $\Lambda$ and the finite functions of $\lambda$ only in MIR scheme.

Owing to the fact that the renormalization group functions depend on the scale $\mu$ only through $\lambda(\mu)$, the renormalization group equation based on MIR can be solved easily.

$$
\Gamma[\phi; m^2, \lambda; \mu^2] = \Gamma[\tilde{\phi}(-t); \tilde{m}^2(-t), \tilde{\lambda}(-t); \mu_0^2] \quad \Rightarrow \quad (2.21)
$$

$$
\Gamma^{(n)}(p, m^2, \lambda; \mu^2) = 
\exp \left[ -n \int_0^{-t} dt' \gamma(\tilde{\lambda}(t')) \right] \Gamma^{(n)}(p, \tilde{m}^2(-t), \tilde{\lambda}(-t); \mu_0^2) , \quad (2.22)
$$

where $p = (p_1, p_2, \cdots, p_n)$, and $\tilde{\lambda}(t), \tilde{m}^2(t), \tilde{\phi}(t)$ are the solutions of the following equations:

$$
\frac{d \tilde{\lambda}(t)}{dt} = \beta(\tilde{\lambda}(t)), \quad \tilde{\lambda}(0) = \lambda \quad (2.23) \\
\frac{d \ln \tilde{m}^2(t)}{dt} = -\gamma_m(\tilde{\lambda}(t)), \quad \tilde{m}^2(0) = m^2 \quad (2.24) \\
\frac{d \ln \tilde{\phi}(t)}{dt} = -\gamma(\tilde{\lambda}(t)), \quad \tilde{\phi}(0) = \phi . \quad (2.25)
$$

In eq. (2.21) - eq. (2.25) $\mu$ is the scale where the initial conditions are defined and $\mu_0$ is a arbitrary reference scale and $t = \ln(\mu/\mu_0)$. When the scale dependence $\tilde{\lambda}$ is solved by eq. (2.23) the scale dependence of the other quantities are
explicitly given. Now it is clear that the determination of the scale dependence of $\Gamma$ reduces to knowing the renormalization group functions $\beta$, $\gamma_m$, $\gamma$ of the considering system.

To see the scaling behavior clearly, the result eq. (2.22) can be rewritten as

$$\Gamma^{(n)}(e^{-t}p, m^2, \lambda; \mu_0^2) = \exp \left[ -(4 - n)t - n \int_0^{-t} dt' \gamma(\bar{\lambda}(t')) \right]$$

$$\times \Gamma^{(n)}(p, e^{2t}\bar{m}^2(-t), \bar{\lambda}(-t); \mu_0^2).$$  \hspace{1cm} (2.26)

This shows how $\Gamma^{(n)}$ behaves when the momentum scale changes from $p$ to $e^{-t}p$. The exponential factor in eq. (2.26) expresses the superficial change of the dimension of the (renormalized) field $\phi$ from ordinal (canonical) dimension 1 to $1 - t^{-1} \int dt' \gamma(\lambda)$ by the effect of the interaction. So the function $\gamma$ is called specially "anomalous dimension". Noting that $\Gamma$ (in other words, Greens function) determines the behavior of a system, eq. (2.26) says that, except the dimensional anomaly, when the momentum scale of a system changes from $p$ to $e^{-t}p$, the system at the new scale $e^{-t}p$ behaves as the system of the scale $p$ but with the change of the mass and coupling, $e^{2t}\bar{m}^2(t)$, $\bar{\lambda}(t)$. Therefore $e^{2t}\bar{m}^2(t)$, $\bar{\lambda}(t)$ are giving the effective mass and coupling of the system in the new scale and they are called "effective mass", "effective coupling constant" or "running mass", "running coupling constant" respectively. Note that when $\mu_0$ is much greater than $\mu$, i.e. $-t$ becomes large, the effective mass drops exponentially so that the behavior in such region can be described well with massless theory. This fact is greatly used in the following sections.

In case of QCD, the $\beta$ function turns out to be able to a negative function of renormalized coupling $g_s$, which can be expressed up to two loops [76, 77] as

$$\beta(g_s, \epsilon) = -\frac{1}{2} \epsilon g_s - \beta_0 g_s^3 - \beta_1 g_s^5 + O(g_s^7)$$  \hspace{1cm} (2.27)

$\beta_0 = \frac{1}{4\pi^2} \left( \frac{11}{3} N_c - \frac{4}{3} T_R n_f \right)$  \hspace{1cm} (2.28)

$\beta_1 = \frac{1}{(4\pi)^4} \left( \frac{34}{3} N_c - \frac{10}{3} N_c n_f - 2C_F n_f \right)$,  \hspace{1cm} (2.29)

where $n_f$ is the number of the active flavors and for later use I kept the term with $\epsilon = 4 - d$ which appears in the dimensional regularization. The eq. (2.27) indicates that, if $n_f < 16$, the running coupling constant in QCD $\bar{g}_s(-t)$
logarithmically approaches to 0 (ultraviolet fixed point, fig. 2.3) as the scale of the system increases \((-t \to \infty)\). Therefore the system asymptotically approaches to the behavior of the free field theory and perturbative treatment is applicable. This behavior which is peculiar to the system of non-Abelian gauge theory \[28\] is called “asymptotic freedom”, which was firstly pointed out in \[26, 27\].

![Figure 2.3: behavior of $\beta$ function in QCD](image)

We often use $\alpha_s(-t) = \left(\bar{g}_s(-t)\right)^2/4\pi$ instead of $\bar{g}_s(-t)$. In the leading order, for simplicity, $\alpha_s(-t)$ becomes

$$
\alpha_s(-t) = \frac{\alpha_s(0)}{1 + 4\pi \beta_0 \alpha_s(0) \ln(\mu_0^2/\mu^2)} \quad (\bar{g}(0) = g) \quad (2.30)
$$

Usually, $\mu$ in eq. 2.30 is replaced by $\Lambda_{QCD}$ which expresses the typical critical scale of the perturbative treatment in QCD.

$$
\ln(\Lambda_{QCD}^2) = \ln(\mu^2) - \frac{1}{4\pi \beta_0 \alpha_s(0)} \quad (2.31)
$$

(Note however we don’t need to always choose $\mu$ as above.) Then eq. 2.30 turns

$$
\alpha_s(-t) \equiv \alpha_s(\mu_0^2) = \frac{1}{4\pi \beta_0 \ln(\mu_0^2/\Lambda_{QCD}^2)} \quad (2.32)
$$

where we simply indicate $\alpha_s(-t)$ as $\alpha_s(\mu_0^2)$. As the results of many experiments, $\Lambda_{QCD} \sim \mu$ is determined as roughly a few hundred MeV \[13\]. From eq. 2.30 we can see that the application of the running coupling $\alpha_s(\mu_0^2)$ serves as a summation of all $(\alpha_s \log(\mu_0^2/\Lambda_{QCD}^2))^n$ terms in perturbative calculation. In case $\mu_0$ is large compared with $\Lambda_{QCD}$, like DIS case, the summation is mandatory to provide faithful predictions in QCD.
2.1.3 Application of RGE to OPE

First I’m going to deal with $\phi^4$ theory again and the simple OPE eq. (2.6) for simplicity and set $A = B = \phi$ because of the complexity with the definition of the renormalized local operators.

At first, OPE eq. (2.6) is equivalent to the following relation between the truncated Green’s functions:

$$\Gamma_{\phi\phi}^{(n)} \left[ \frac{q}{2} + k, \frac{q}{2} - k, p, m^2, \lambda; \mu^2 \right] = \sum_i C_i [k, m^2, \lambda; \mu^2] \Gamma_{O_i}^{(n)} [q, p, m^2, \lambda; \mu^2]$$

$$\langle 0 | O(x) \phi(z_1) \cdots \phi(z_n) | 0 \rangle^{1PI} = \left[ \prod_{i=1}^n \int dy_i \Delta_F (z_i - y_i) \right] \Gamma_{\phi}^{(n)} [x, y, m^2, \lambda; \mu^2],$$

where $y = (y_1, \ldots, y_n)$ and $\Delta_F$ is the Feynman propagator of $\phi$. For the definition of $\Gamma_{\phi}$, which is equivalent to the definition of the local operator $O_i$, the additional renormalization conditions are needed. Now consider to apply RGE eq. (2.17) or equivalently

$$(\mathcal{D} - n\gamma(\lambda)) \Gamma_{\phi\phi}^{(n)} [p; m^2, \lambda; \mu^2] = 0 \quad (2.34)$$

$${\mathcal{D}} = \mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma_m(\lambda) m^2 \frac{\partial}{\partial m^2} \quad (2.35)$$

to OPE eq. (2.33)

Applying $\mathcal{D}$ to the left-hand side of eq. (2.33)

$$\mathcal{D} \Gamma_{\phi\phi}^{(n)} = (n - 2) \gamma(\lambda) \Gamma_{\phi\phi}^{(n)}$$

$$\Rightarrow \quad \mathcal{D} \Gamma_{\phi\phi}^{(n)} = \sum_i (n - 2) \gamma(\lambda) C_i \Gamma_{O_i}^{(n)}. \quad (2.36)$$

Then next, apply $\mathcal{D}$ to the right-hand side. In general, for the renormalization of the local operator $O_i$ to satisfy its renormalization conditions, many counter terms $Z_{ij}(O_0)_j$ (bare operators with peculiar renormalization constants $Z_{ij}$) are required. The operators $(O_0)_j$ appearing in the counter terms for $O_i$ are restricted to the operators which have the same quantum number and Lorentz transform as $O_i$ and the canonical dimension less than that of $O_i$. But those are arbitrary except for the constraints. For example, for the renormalization of the local operator $\phi^4(x)$, the following operators can emerge in the counter terms:

$$\phi_0^4, \phi_0^2, \partial_\mu \phi_0 \partial^\mu \phi_0, \phi_0 \Box \phi_0, \cdots \quad (2.37)$$
Therefore the renormalized local operator $O_i$ can be expressed formally as

$$O_i = \sum_{d_j \leq d_i} \tilde{Z}_{ij}(\lambda_0, A, \mu, m^2_0) (O_0)_j$$ \hspace{1cm} (2.38)

Here the vector $(O_0)_j$ aligns in order of the canonical dimension and $\tilde{Z}_{ij}$ is not 0 when $d_j \leq d_i$. This mixing of operators which appears in the renormalization of local operators is called “operator mixing”. With this notation of $O_i$,

$$\Gamma^{(n)}_{O_i} = \sum_j \tilde{Z}_{ij} Z_{3}^{n/2} \Gamma^{(n)}_{0O_j}[q, p, m^2_0, \lambda, \Lambda].$$ \hspace{1cm} (2.39)

Therefore

$$(\mathcal{D} - n\gamma(\lambda))\Gamma^{(n)}_{O_i} + \sum_j \tilde{\gamma}_{ij}(\lambda, \mu, m^2)\Gamma^{(n)}_{O_j} = 0,$$ \hspace{1cm} (2.40)

$$\tilde{\gamma}_{ij} \equiv - \sum_k \left( \mu \frac{\partial \tilde{Z}_{ik}}{\partial \mu} \right)_0 \tilde{Z}_{kj}^{-1}$$ \hspace{1cm} (2.41)

Then the application of $\mathcal{D}$ to the right-hand side of eq. 2.33 leads

$$\mathcal{D} \sum_i C_i \Gamma^{(n)}_{O_i} = \sum_i \left[ (\mathcal{D}C_i) \Gamma^{(n)}_{O_i} + C_i \mathcal{D}\Gamma^{(n)}_{O_i} \right]$$ \hspace{1cm} (2.42)

$$= \sum_i \left[ \mathcal{D}C_i + C_i n\gamma(\lambda) \right] \Gamma^{(n)}_{O_i} - \sum_{ij} C_i \tilde{\gamma}_{ij} \Gamma^{(n)}_{O_j}$$ \hspace{1cm} (2.43)

$$= \sum_i \left[ \mathcal{D}C_i + C_i n\gamma(\lambda) - \sum_j C_j \tilde{\gamma}_{ji}(\lambda) \right] \Gamma^{(n)}_{O_i}.$$ \hspace{1cm} (2.44)

Equating eq. 2.36 and eq. 2.44 and taking the coefficient of $\Gamma^{(n)}_{O_i}$,

$$(\mathcal{D} + 2\gamma(\lambda) - \tilde{\gamma}^T(\lambda, \mu, m^2)) C[k, m^2, \lambda; \mu^2] = 0,$$ \hspace{1cm} (2.45)

where $C$ is the vector composed of $C_i$ and $\tilde{\gamma}^T$ is the transposed matrix of $\tilde{\gamma}_{ij}$. The answer of eq. 2.45 is given, like the case of eqs. 2.17 and 2.22 as

$$C[k, m^2, \lambda; \mu^2] = T_t \exp \left( - \int_0^{-t} dt' \left[ 2\gamma(\tilde{\lambda}(t')) - \tilde{\gamma}^T(\tilde{\lambda}(t'), \tilde{m}^2(t'), \mu_0) \right] \right)$$

$$\times C[k, \tilde{m}^2(-t), \tilde{\lambda}(-t); \mu^2_0],$$ \hspace{1cm} (2.46)

where $T_t$ defines the order of the matrix so that the matrix with smaller $t'$ comes earlier.
Therefore, it now turns out that the short distance scale \( k \) dependence of OPE can be expressed simply by eq. \( 2.46 \) and, especially for the case of the theory with asymptotic feature like QCD, it can be calculated well by the perturbative method with massless theory in case \( \mu_0 \sim k \gg \mu \sim m^2 \) as mentioned in the previous section.

Now let us turn to the DIS case. In eq. \( 2.45 \) the term \( 2\gamma \) is reflecting \( A = B = \phi \). When \( A \) and \( B \) are other (local) operators, the \( \gamma \)'s are substituted with the corresponding (matrix) \( \gamma \)'s. The current \( J \) as the local operator does not receive the renormalization correction because of its physical meaning. So the renormalization constant \( Z_J \) must be 1 and the anomalous dimension of the current \( J \) turns out to be zero. By the application of the light-cone OPE, eq. \( 2.8 \) and the dispersion relation based on Cauchy theorem in complex \( 1/x \) plane to get the result in the DIS kinematic region \( 0 < x \leq 1 \) where \( x \) is the Bjorken \( x = -q^2/P \cdot q \), we can obtain finally the following results:

\[
\int_0^1 dx x^{n-1} W(x, Q^2) = \sum_i A^i_n \tilde{C}^i_n(Q^2) \tag{2.47}
\]

\[
\tilde{C}^i_n(-q^2)q_{\mu_1} \cdots q_{\mu_n}(-q^2/2)^{-n} = \int d^4\xi e^{iq\cdot\xi} \xi_{\mu_1} \cdots \xi_{\mu_n} C^i_n(\xi^2) \tag{2.48}
\]

\[
\langle P|O_{\mu_1 \cdots \mu_n}^i|P \rangle = A^i_n P_{\mu_1} \cdots P_{\mu_n}, \tag{2.49}
\]

where \( W \) is some quantity given from \( W^{\mu\nu} \) in \ref{eq:2.4} with appropriate contractions of its Lorentz indices which corresponds to structure functions defined in the next chapter and \( A^i_n \)'s are constants which contain all long distance information. The behavior of dimensionless quantity \( \tilde{C}^i_n(Q^2) \) is given by virtue of RGE like eq. \( 2.46 \) as

\[
\tilde{C}^i_n \left( \frac{Q^2}{\mu^2}, m^2, g \right) = \tilde{C}^i_n \left( \frac{Q^2}{\mu_0^2}, 0, \tilde{g}(t) \right) T_\nu^T \exp \left[ - \int_0^t dt' \tilde{\gamma}_i^T \tilde{\gamma}_O(t') \right], \tag{2.50}
\]

where I neglected mass term on the right-hand side assuming \( Q^2 \sim \mu_0^2 \gg \mu^2 \sim m^2 \) and set \( t = \ln(\mu_0/\mu) \) unlike eq. \( 2.46 \). The detailed derivation can be found in \[73, 78\].

Now the short distance \( (Q^2) \) behavior of the structure function reduces to the calculation of \( \gamma_O \). With the massless theory, the operator mixing is furthermore restricted in operators with the same dimension. For example, in massless QCD twist 2 operators which can mixed are eq. \ref{eq:2.10} and eq. \ref{eq:2.11} and their \( \gamma_{ij} \) to the one-loop (leading) order can be calculated in Feynman gauge by the diagrams figs.\ref{fig:2.4} and \ref{fig:2.5}.  

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\[ \bullet O^F = Z_{FF} O^F_0 + Z_{FV} O^V_0 \]

Figure 2.4: the operator mixing of twist 2 operators for \( O^F \)

\[ \bullet O^V = Z_{VF} O^F_0 + Z_{VV} O^V_0 \]

Figure 2.5: the operator mixing of twist 2 operators for \( O^V \)

In general \( C_i^o \) and \( \gamma_{O^i_n} \) are expanded perturbatively as

\[ \gamma_{O^i_n} = \left( \frac{\alpha_s}{4\pi} \right) \gamma_{0n} + \left( \frac{\alpha_s}{4\pi} \right)^2 \gamma_{1n} + \cdots \] (2.51)

\[ \tilde{C}_n \left( \frac{Q^2}{\mu_0^2} \right) = c_{0n}^i + \left( \frac{\alpha_s}{4\pi} \right) c_{1n}^i + \cdots, \] (2.52)

where \( c_{jn}^i \) is in general a function of \( Q^2/\mu_0^2 \). The structure functions \( W_i \) in eq. 2.47 are calculated by taking the first terms of eqs. 2.51 and 2.52 for the leading (one-loop) order (LO) and taking up to the second terms for the next-leading (two-loop) order (NLO). As indicated in the proceeding sections, these corrections surely gives the perturbative correction to the simple parton model.

If we simply set \( Q^2 = \mu_0^2 \), consider the contribution only from non-mixing
(non-singlet) operators, then eq. 2.50 becomes in LO accuracy as

\[ \tilde{C}_n \left( \frac{Q^2}{\mu^2}, m^2, g \right) = c_{0n}^i \exp \left[ A \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d(\ln \alpha'_s) \right] \]

\[ = c_{0n}^i \left[ \alpha_s(Q^2) \right] A \]

\[ = c_{0n}^i \left[ 1 + 4\pi \beta_0 \alpha(\mu^2) \ln(Q^2/\mu^2) \right]^{-A} , \quad (2.53) \]

where \( \beta_0 \) is from eq. 2.28. (Remember that \( \alpha_s(\mu) = \alpha_s(0) \) in eq. 2.30 and note RGE serves as a summation of \( \log(Q^2/\mu^2) \) also here.) Then from eq. 2.53 we can see \( \tilde{C}_n^i \) scales as \( \ln(Q^2)^{-A} \) for \( Q^2 \to \infty \). This behavior gives logarithmic correction to naive OPE, eq. 2.39 and predicts the logarithmic breaking (or correction) of Bjorken scaling which is expressed as no interaction limit. Historically it became an definite evidence of validity of QCD as strong interaction along with the observation of the breaking. This successful explanation of (unpolarized) DIS with the QCD radiative corrections was first made in \[79, 80\] with the above tools. This explanation is well reviewed in \[81, 82, 78\]. It was extended in NLO in \[83, 75\].

So far the radiative corrections to parton model can be derived well in case of DIS with the language of OPE and RGE. However this method cannot be extended to more general high energy process, like hadron detection in \( e^+ - e^- \) collision, which will be discussed in the next chapter. This is because the limitation of the application of OPE to those processes. Therefore, if we want to go further to investigate the pQCD radiative corrections in the general processes standing on the success of DIS description, we have to grab a method which has more general basis than OPE to factorize the short and long behavior. Noting that, in OPE, we exploit short distance behavior actively as coefficient function \( C \), we come to an idea to extract long distance behavior instead from Feynman diagrams expressing required interaction. In radiative correction, the long distance scale behavior comes from infrared (IR) divergences. Therefore we can remove the long distance behavior by eliminating the infrared nature of the interaction. In case some typical hard scale \( Q \) of the interaction is much larger than the masses of quarks, like DIS, these masses can be neglected in radiative correction calculations as indicated in the above discussion. As we will see in the next section, these massless quarks serves a source of a type of IR divergence, collinear divergence, or mass singularity. Therefore the collinear divergences which newly appear in high energy interaction would become a sign of the existence of a hard scale interaction. This fact can be made use of for factorization of long distance
behavior by eliminating those mass singularities in diagrams of those hard interactions. The method based on this idea is explained in the following sections and as a result it surely keeps the factorization form of eq. 1.5 in an appropriate gauge. This is the alternative method for factorization, called mass factorization [84, 85]. Because this method is based on more intuitive diagrammatic treatment, it is applicable to general processes.

2.2 Properties of Infrared Divergences

First, I would like to briefly comment on general properties of infrared (IR) divergences and provide the clear picture of the collinear divergence. For its purpose, it would be convenient to consider the 2 → 3 process of $e^+e^- \rightarrow q\bar{q}g$ (3 jet process in $e^+e^-$ collision) with massless quarks and the square of its center mass energy $s = q^2$, like fig. 2.6. The massless condition is justified in $s \gg m_q^2$.

![Figure 2.6: 3 jet process in $e^+e^-$ collision](image)

Considering the process as 2 → 2 process with virtual photon and the partons, the differential cross section on fractional energy of quarks can be described as

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} ,$$

where $\sigma_0$ is the two jet cross section, $4\pi\alpha^2_e/s \times \sum_q Q_q^2$ and $x_i (i = 1, 2, 3)$ is the fractional energy of the particle $i$ defined by $2E_i/\sqrt{s} > 0$. The energy conservation implies $\sum_i x_i = 2$. To see the possible kinematic range of $x_i$, define $\theta_{ij}$ as the angle between the momenta of partons $i$ and $j$. Then we can relate these angles to $x_i$ by

$$(p_1 + p_3)^2 = m_1^2 + 2 p_1 \cdot p_3 = (q - p_2)^2 = s - 2q \cdot p_2 + m_2^2 ,$$

$$\rightarrow 2E_3(E_1 - p_1 \cos \theta_{31}) = q^2(1 - x_2) ,$$

(2.55) (2.56)
where quark masses are kept finite for later use. In the limit of massless, we obtain

\[ x_3 x_1 (1 - \cos \theta_{31}) = 2 (1 - x_2), \quad (2.57) \]
\[ x_1 x_2 (1 - \cos \theta_{12}) = 2 (1 - x_3), \quad (2.58) \]
\[ x_2 x_3 (1 - \cos \theta_{23}) = 2 (1 - x_1). \quad (2.59) \]

Then we can get \( x_i < 1 \). With the momentum conservation \( \sum_i x_i = 2 \), the arrowed region for \((x_1, x_2)\) turns out to be a triangle, as shown in fig. 2.7.

Figure 2.7: allowed kinematic range for \( x_1 \) and \( x_2 \)

It is important to note that the edges of the region correspond to collinear and soft configurations: the edges of \( x_i = 1 \) accord with two partons being collinear, \( \theta_{ij} \to 0 \Leftrightarrow x_k \to 1 \), and the corners of \( x_i = 0 \) with one parton momentum being soft, \( p_i^\mu = 0 \), fig. 2.8.

Figure 2.8: kinematic configuration in the \((x_1, x_2)\) region

Now we can see the cross section eq. 2.54 has singularities in those configurations: collinear singularities,

\[ (1 - x_1) \to 0, \quad (\text{partons 2 and 3 are collinear}) \quad (2.60) \]
\[ (1 - x_2) \to 0, \quad (\text{partons 1 and 3 are collinear}) \quad (2.61) \]
and soft singularities of $x_3 \to 0$,

\[ (1 - x_1) \to 0, \ (1 - x_2) \to 0, \ \frac{(1 - x_1)}{(1 - x_2)} \sim \text{const}. \quad (2.62) \]

From eq. 2.56 we can observe the collinear singularities appear only if the quarks are regarded as massless. So these are also called mass singularities. The soft divergence also stems from massless nature of gluons and quarks. Then, because of the existence of these singularities, the total cross section of eq. 2.53 diverges when the integration is taken. To see the nature of the divergence, let us rewrite the cross section in a way that it displays the collinear singularity at $\theta_{31} \to 0$ and the soft singularity at $E_3 \to 0$ obviously.

\[
\frac{d\sigma}{dE_3\,d\cos\theta_{31}} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \frac{f(E_3, \theta_{31})}{E_3(1 - \cos\theta_{31})},
\]

where $f(E_3, \theta_{31})$ is a function finite both in $E_3 \to 0$ and $\theta_{31} \to 0$. limits. Then we see the integrations around the limit gives logarithmic divergence both for soft and collinear cases. With some cut-off $\Lambda_{IR} \sim m_q$, these typically appear as $\log(s/\Lambda_{IR}^2)$. These singularities are known as infrared (IR) divergences appearing in general everywhere (also in loops) in Feynman diagrams. Therefore these divergences must be treated as well for quantitative predictions based on field theories as ultraviolet (UV) divergences with the renormalization. As UV divergences originate from those in large momentum region, i.e. short distance, the IR divergences are of those of long distance nature.

To see the long distance nature of the infrared divergences, it is convenient to define null plane coordinate, $p^\mu = (p^+, p^-, p^1, p^2) = (p^+, p^-, \mathbf{p}_T)$ where

\[
p^\pm = \frac{(p^0 \pm p^3)}{\sqrt{2}}. \quad (2.64)
\]

Then $p^2 = 2p^+p^- - \mathbf{p}_T^2$ and

\[
d^4p = dp^0 dp^+ dp^- d^2p_T = -dp^+ dp^- d^2p_T. \quad (2.65)
\]

Thus for a particle on its mass shell, $p^-$ becomes

\[
p^- = \frac{\mathbf{p}_T^2 + m^2}{2p^+}. \quad (2.66)
\]

For a particle with large momentum compared with its mass in the $+p_3$ direction and limited transverse momentum, $p^+$ is large and $p^-$ becomes
null plane axes in momentum space

Figure 2.9: null plane axes in momentum space

small. We (can) often choose the plus axis so that a particle or group of particles of interest have large \( p^+ \) and small \( p^- \) and \( p_T \), fig. 2.9.

Then in the 3 jet process, define \( k^\mu = p_1^{\mu} + p_3^{\mu} \), like fig. 2.10 and take the null plane coordinates with \( k^+ \) large and \( k_T = 0 \). Then \( k^2 = 2k^+k^- \) where

\[
k^- = \frac{p^2_{3,T}}{2p_1^+} + \frac{p^2_{3,T}}{2p_3^+}.
\]

In the IR configurations of eqs. 2.60 - 2.62 we can see \( k^- \to 0 \) when we take \(+k_3\) axis as \( p_i : p^2_{3,T} \to 0 \) with fixed \( p^+_{1,3} \) means \( p_3 \) is going to collinear to \( p_1 \), and \( p^2_{3,T} \to 0 \) with \( p^+_{1,3} \propto |p^2_{3,T}| \) gives \( p_{1,3} \) going soft. Therefore, in the region near the IR configurations, we can take the coordinates that \( k^+ \sim |p_i| \) becomes large, \( k^- \) goes to zero and then \( k^2 \) reaches to 0. Noting that Fourier transform of \( k^\mu \) to space-time \( x^\mu \) is given through

\[
k \cdot x = k^+x^- + k^-x^+ - p_Tx_T,
\]

That configuration in IR region turns to that of \( x \) with small \( x^- \) and large \( x^+ \) as in fig. 2.10. Thus we can certainly see the IR singularities arise from the interactions that happen a long time after the creation of the initial quark-antiquark pair. Note that space-time configuration of the creation is characterized as \( 1/\sqrt{s} \to 0 \), i.e. short distance. IR region is surely contrast to the short distance behavior.

These IR divergences has to be handled well as those of UV for quantitative predictions especially in perturbative expansion because IR divergences can spoil the prediction as UV case. Actually the IR divergences grows up in high energy interaction as \( \log(Q^2/m_q^2) \). Then re-summation is needed as UV case with RGE. We will see this in the next section. In QED with massive leptons, this problem (only with soft divergence from photon) is avoided by
the summation of all the contributions of degenerate final state configurations, known as Bloch-Nordsieck theorem [86, 87]. Noting that we cannot distinguish observed leptons with that accompanied with emitted indefinite number of soft photons (degenerate states) in sense of its detection, we arrives at the summation of those soft emission processes at the same time for the prediction of a observable (physical) quantity. Bloch-Nordsieck theorem tells us that the physical quantity becomes infrared safe in all orders of perturbative expansion after the summation of the final degenerate states, fig. 2.11 The IR contribution coming from loops exactly cancels with those from the degenerate states. The cancellation in the summation in all orders are well summarized in textbooks [73, 78] or detailed papers [88, 89].

However in QCD case, there also exists the collinear divergence even in massive quarks through gluon self-couplings. In spite of this situation, it is believed that the infrared divergences in QCD can cancel out in all orders if the physical quantities are considered. This is because the cancellation of collinear and soft divergences becomes generally possible when we take
summation of not only degenerate states with collinear and soft particles in final states but also those in initial states, which is known as Kinoshita-Lee-Nauenberg theorem [90,91], fig. 2.12. Thus it serves as the generalization of the Bloch-Nordsieck theorem. The proof can be found with a sophisticated manner in [92].

Figure 2.12: summation of the soft and collinear degenerate states in final and initial states

As an example of the infrared safe quantity, there is the quantity known as R ratio in $e^+e^-$ collision defined as the ratio between the total cross section of hadron channel in the final state and that of lepton channel,

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \text{leptons})}.$$  \hspace{1cm} (2.69)

Because it includes all degenerate states in the final states (both in numerator and denominator) as the full integration of all the final states. The jet definition using $\delta - \epsilon$ prescription [93] is also infrared safe quantity. we can see another examples of IR safe quantities in $e^+e^-$ collision in [94].

Here, it would be worth noting the cutting rules related to IR cancellation (also for the preparation to the discussion in the next section.) If we integrate of all the kinematic region of the final states like in case of R ratio or the total cross section, it is convenient to use cut diagrams (cutting rule) [95]. It also has another advantage of making the cancellation of IR divergence manifest [96,97]. The cutting rule is a general method to calculate the imaginary part of any Feynman diagrams using the general analytic property of the diagrams, i.e. discontinuity appearing in integration over physical on-shell region. The “rule”s are the following [95]:

- Cut through the diagram in all possible ways such that the cut propagators can simultaneously be put on shell.
For each cut, replace the denominator of the propagator, \( \frac{1}{(p^2 - m^2 + i\epsilon)} \), by \(-2\pi i\delta(p^2 - m^2)\) and then perform the loop integration.

Sum the contributions of all possible cuts.

The imaginary part or equivalently the physical discontinuity of the diagrams has special significance for the forward scattering amplitudes. The imaginary part of the forward amplitudes becomes equivalent to a square of the cut diagrams with the full integration of the phase space of the physical intermediate (final) state by the optical theorem, like fig. 2.2 or more generally;

\[
-\ii [\mathcal{M}(a \to b) - \mathcal{M}^*(b \to a)] = \sum_f \int d\Pi_f \mathcal{M}^*(b \to f)\mathcal{M}(a \to f),
\]

(2.70)

where \( \mathcal{M} \) is the arbitrary invariant matrix element and \( a, b, f \) are any on-shell states. Using the cutting rules, it becomes possible to prove the optical theorem, eq. (2.70) diagrammatically to all orders in perturbation theory. What \( [96, 97] \) proved is that the cancellation of the IR divergence occurs between the diagrams given by a cut. A set of diagrams for which the cancellation of the IR divergence takes place is generated by cutting of the same original diagram. If the original diagram is IR safe in sense of dimensional counting, the IR divergences existing in the each of the cut diagrams should compensate each other to make the contribution of the cut diagrams through the optical theorem eq. (2.70) finite. For example in \( e^+e^- \) inclusive cross section up to the next next leading order, the cut diagrams are given as fig. 2.13. As discussed in the next section, the forward amplitude of a virtual photon, fig. 2.13 is IR divergence free. (note that it is gauge independent.) Therefore the contribution from diagrams given by a cut and the optical theorem does not have IR singularity and then total cross section as the sum of the contributions is obviously IR safe.

In case of massless case accompanied with the potential danger of collinear divergence, the cutting is generalized to include the cut between initial and final lines \( [90, 98] \).

In spite of the successful interpretation of the physical IR quantity, this does not work for the parton model description, which is identical to lowest twist contribution and dominates in DIS kinematic region. The parton model with radiative corrections can be described as fig. 2.14. This is because the contribution from the initial state degenerate states (involving more than one active parton per initial hadron) serves as higher twist contribution and these contribution declines by powers of the large invariants.
Figure 2.13: cut diagrams in $e^+e^-$ inclusive cross section

Figure 2.14: parton description of DIS with radiative corrections
$Q^2$ compared with that of the lowest twist. Therefore the parton model description with radiative corrections in general has IR divergences. What I’m going to show in the next section is that the IR divergences, especially the collinear divergences, can be systematically eliminated from a short distance behavior as a factor after the summation of the divergences with RGE which is inevitable for high energy interactions. The IR divergent factor is absorbed into the definition of parton distribution functions. This is nothing but the diagrammatic expression of the factorization of the long and short distance behaviors.

For making it clear of the meaning of the factorization of IR divergence in the parton model, it would be nice to see the similarity of the factorization with the renormalization like eqs. 2.13 and 2.14, so to say the factorization of UV divergence. It is expressed as fig. 2.15.

![Figure 2.15: Correspondence between Renormalization and Factorization](image)

The cut-offs of $\Lambda_{UV}$ and $\Lambda_{IR}$ needed for finite predictions from local field theories or equivalently the regulators of $\epsilon_R$ and $\epsilon_F$ in dimensional regularization are absorbed into the definition of the renormalized quantities of field, running coupling constant and running masses for the renormalization and the definition of the parton distribution functions for the factorization with the
help of RGE. The RGE is one of the key issues to relate the physics at different scales and give the effective predictions avoiding the potential danger to spoil the perturbative expansion. Those redefinitions in both energy limits come from the limitation of the application of the local field theory with the perturbative method in those limits. For the later convenience, let the analogy more concrete in the table [2.1]. The quantities in the factorization column will be given in the next section.
Table 2.1: analogy between renormalization and factorization quantities

<table>
<thead>
<tr>
<th>A: Bare Green Func.</th>
<th>$\Gamma_0^n(\lambda_0, m_0, \ldots, \Lambda_{UV})$</th>
<th>IR factorization</th>
</tr>
</thead>
<tbody>
<tr>
<td>B: Ren. Constants</td>
<td>$Z_i(\mu_R, \epsilon_R)$</td>
<td>Partonic X-sec</td>
</tr>
<tr>
<td>C: Ren. Green Func.</td>
<td>$\Gamma^n(\mu_R) = \Gamma_0^n/Z_i^n$</td>
<td>Hard X-sec</td>
</tr>
<tr>
<td>D: Anomalous Dim.</td>
<td>$\gamma(\mu_R) = \frac{\mu}{Z_s} \frac{dZ_s}{d\mu_R}$</td>
<td>Splitting Func.</td>
</tr>
<tr>
<td>E: UV Free Quant.</td>
<td>$\lambda(\mu_R) = \lambda_0/Z$, $\Gamma_R^n(\lambda, \ldots, \mu_R)$</td>
<td>IR Free Quant.</td>
</tr>
</tbody>
</table>

- A : divergent; but independent of scale $\mu_{F,R}$;
- B : divergent; scale $\mu$ dependent; absorbs all the UV or IR divergences;
- C & D : finite; scale dependent; D controls the $\mu$ dependence of E;
- E : theoretical prediction; $\mu$ independent to all orders
  considering D effect; but dependent at finite order;

- “Renormalization” is a factorization of UV divergences
  and ”Factorization” is a renormalization of IR divergence –
2.3 Factorization of Collinear Divergences (Mass Factorization)

In this section, I’m going to show the factorization of collinear divergences (mass factorization) in the diagrams of parton picture, like fig. 2.14, based on the discussion in [84, 85]. (it would be helpful to refer also to [99].) This factorization process, sometimes referred as mass factorization, can be done systematically. Indeed I’m concentrating only on DIS case, the extension to the other processes seems obvious because it relies on general diagrammatic treatment.

By the cutting rule introduced in the previous section, what we like to investigate for the matrix element squared of the DIS process is the diagram like fig. 2.16 as forward amplitude of initial quark or gluon with light-like momentum $p$ and virtual photon with $q$. In the following, the cutting rule on the diagram is understood. (Actually most of part of discussion is based only on the dimensional counting. In that sense, we don’t need to assume the cut everywhere. We can understand the cut requirement only for the quantity related to the cross section, like DGLAP equation.) The first step for the systematic factorization is the decomposition of the squared matrix element into two particle irreducible (2PI) kernels of $C_0$ and $K_0$, fig. 2.16. (These kernels are considered to be an assembly of 2PI diagrams of arbitrary orders.) For simplicity let me consider all the connecting lines are quark line.

The 2PI amplitude is the counter object to two particle reducible (2PR) amplitude which is defined in case that a couple of external lines connected by a cut, i.e. $p_i$ and $-p_i$ ($p$ channel), can be separated from the other external lines by cutting precisely two internal lines, fig. 2.17. The key observation for the factorization is that the 2PI amplitudes in a physical gauge, axial gauge, are IR divergent free (for both collinear and soft divergences) as long as the external legs are all on-shell. While if a couple of legs, say $k$ channel, in a 2PI diagram are off-shell and integrated to connect to other sub-diagram which makes the diagram 2PR as a whole, it causes logarithmic divergence on the edge of its on-shell limit, i.e. $\log(k^2)$, fig. 2.17. (Remember $k^2$ became 0 in IR limit in the previous section.) Even after the regularization of such logarithmic IR divergences coming from the connection integral between 2PI kernels by some method, the logarithmic contribution becomes large in high energy limit exactly like the case of running coupling constant. The contributions from $n$th order might comparable with $n-1$th or $n$th order contribution. Therefore the re-summation of these logarithms are necessary.
The fact urges us to consider the ladder structure comprehensive diagrams composed of series of 2PI kernels like fig. 2.16.

The proof of the finiteness of the 2PI amplitude was given in [85] based on the simple dimensional counting. In the proof, the application of the physical axial gauge plays a crucial role. In appendix B I shortly summarised the property of the axial gauge, specially light-cone gauge I’m going to apply. The finiteness of the 2PI amplitude is a graphical restatement of the fact that the divergence associated with a particular physically allowed sub-process comes from the square of the amplitude describing the sub-process in the physical axial gauge. In case of 3 jet process in $e^+e^-$ collision discussed in the previous section, this means that, in the axial gauge, the collinear and soft divergences according to the integration on the phase space of one of the parton, say gluon this case, are coming independently from the square of the diagrams (2PI), i.e. fig. 2.6 and the one with gluon attaching to antiquark line, and those interference diagram (2PR) are free from those divergences, fig. 2.18. Note that when we take integration on the phase space of all of the three legs, the diagram reduces to 2PI amplitude with no external parton legs. It is nothing but the IR absence in the diagram for the total cross section discussed in the
end of the previous section. In OPE language, the finiteness of 2PI kernel $C$ is the diagrammatic expression of the renormalization part with the almost local (light-cone) operator in the proof of OPE [71]. Historically this notation of the similarity urges to the application of pQCD also to time-like process (one hadron inclusive production in $e^+e^-$ collision) [100], which does not accord with OPE formalism. It was named cut vertex method [101].

Now we like to handle the IR divergences in the ladder type diagrams under our control to eliminate all of these divergences. Noting that IR divergences are coming from the integration on the connecting legs between 2PI kernels and the divergences originate from the on-shell limit of the legs, let me introduce a projection operator $\mathcal{P}$ which works on a channel of legs of 2PI kernel $K$ and project the momenta of the legs of the channel to on-shell physical momentum state. After the implementation of the projection operator, it is expected that the IR divergence only exist in the integration on the projected states $\mathcal{P}K$ and the remnant of the projection $(1 - \mathcal{P})K$ does not concede IR divergence in the integration. If $\mathcal{P}$ is properly chosen, the squared matrix element $M$ as the ladder expansion (LE) of 2PI kernel can be written in demanded factorized form. First, LE can be expressed as

$$M = C_0 \left( 1 + K_0 + K_0^2 + \cdots \right) \equiv C_0 \frac{1}{(1 - K_0)} \equiv C_0 \Gamma_0 \, .$$

(2.71)

Here the perturbative series for $M$ has been rearranged in a way that the factorization starts to be visible: $C_0$ (as a 2PI kernel) is finite while $\Gamma_0$ contains
Figure 2.18: IR contribution from 3 jet process

all mass singularities. Note $C_0$ and $\Gamma_0$ are connected by the integrations on the legs of $C_0$. Applying the projection operator $\mathcal{P}$, eq. 2.71 can be re-expressed as

\[
M = C_0 \left( 1 + \sum_{i=1}^{\infty} K_0^{i-1} \mathcal{P} K_0 + \sum_{i=1}^{\infty} K_0^{i-1} (1 - \mathcal{P}) K_0 \right) \quad (2.72)
\]

\[
C_0 \left( 1 + \sum_{i=0}^{\infty} K_0^i (1 - \mathcal{P}) K_0 \right) + M \mathcal{P} K_0 , \quad (2.73)
\]

\[
\implies M (1 - \mathcal{P} K_0) = C_0 \left( 1 + \sum_{i=0}^{\infty} K_0^i (1 - \mathcal{P}) K_0 \right) . \quad (2.74)
\]

In the last step, we could eliminate the IR divergence from the integral on the first (right most) kernel $K_0$ on the right-hand side. Using this manipulation recursively, we can get the final form

\[
M (1 - \mathcal{P} K) = C_0 \frac{1}{1 - (1 - \mathcal{P}) K_0} , \quad (2.75)
\]

where the modified kernel $K$ is defined as

\[
K = \frac{K_0}{1 - (1 - \mathcal{P}) K_0} . \quad (2.76)
\]
Here and there \(1/(1 - (1 - \mathcal{P}) K_0)\) is defined as the series expansion in which \((1 - \mathcal{P})\) acts on the full expression on the right,

\[
\frac{1}{1 - (1 - \mathcal{P}) K_0} \equiv 1 + (1 - \mathcal{P}) K_0 + (1 - \mathcal{P})(K_0(1 - \mathcal{P}) K_0) + \cdots. \tag{2.77}
\]

Then we notice that the right hand side of eq. 2.75 does not provide IR divergence. Eq. 2.75 can be formally rewritten as

\[
M = \left(C_0 \frac{1}{1 - (1 - \mathcal{P}) K_0}\right) \left(\frac{1}{1 - \mathcal{P} K}\right). \tag{2.78}
\]

If we define \(1/(1 - \mathcal{P} K)\) as a series expansion in which \(\mathcal{P}\) acts only on the neighbouring \(K\) on the right,

\[
\frac{1}{1 - \mathcal{P} K} = 1 + \mathcal{P} K + (\mathcal{P} K)(\mathcal{P} K) + \cdots, \tag{2.79}
\]

\[
= 1 + \mathcal{P} K_0 + \mathcal{P} K_0(1 - \mathcal{P}) K_0 + \cdots. \tag{2.80}
\]

we can directly check the formal form of eq. 2.78 is equivalent to the original LE eq. 2.71. In the final form of eq. 2.78, the factorization of IR divergence becomes obvious: the first factor has no IR divergence and the second contains all the IR divergence in the LE and these factors are coupled by a integral on legs connecting them.

In the following, I’m going to concentrate on elimination of collinear divergences in the light-cone gauge following [84]. Actually the application of the light-cone gauge with the way of [84] has subtlety for the treatment of \((p \cdot n)^{-1}\) factor in gluon propagation. In the method, the anomalous dimension which should not be dependent on gauge as \(\beta\) showed a explicit gauge dependent term, originating from the pole related to the \((p \cdot n)^{-1}\) factor, called spurious pole. To avoid it, they took somewhat intuitive remedy for it. Properly, the method given in [102, 103] should be taken, but its makes the considered diagram increase. (Review on these points can be found in [104].) However, still there is a fact that even in the method of we can obtain correct result up to NLO for polarized case and unpolarized case [105, 106]. (The full consistency of the “phenomenological rule” in [84] with the proper method seems for me to be still under investigation.) For the systematic elimination of soft divergences can be found in [107, 108, 109, 110, 111] for example. The summation of soft divergence becomes significant in case that there are several but different enough high energy scales in a interaction under consideration, like \(W^2 \gg Q^2 \gg \Lambda_{QCD}\) (small x region) in DIS case. For current interest of this analysis, we are going to concentrate on the collinear
To see the function of the projection operator for the isolation of the collinear divergences concretely, let me look bit closer to the actual calculation process. let us consider a 2PI kernel $K_0(k, p)$ with on-shell incoming massless quark line with momentum $p^\mu = (p, 0, p)$, i.e., infinite momentum frame \[112\], and outgoing quark line with momentum $k^\mu$ which is integrated for connecting to an upper 2PI kernel, like in fig. \[2.16\]. For that purpose, it is convenient to decompose $k^\mu$ into the form of the Sudakov decomposition \[113, 114\]:

\[
\begin{align*}
k^\mu &= x p^\mu + \alpha n^\mu + k_{\perp}^\mu = k^\mu_{\parallel} + \alpha n^\mu + k_{\perp}^\mu, \\
 n &= \left( \frac{p \cdot n}{2p}, 0, -\frac{p \cdot n}{2p} \right), \\
x &= \frac{k \cdot n}{p \cdot n}, \\
k_{\perp} &= (0, k_{\perp}, 0), \\
\alpha &= \frac{k^2 - k_{\perp}^2}{2x p \cdot n} = \frac{k^2 + k_{\perp}^2}{2x p \cdot n},
\end{align*}
\] (2.81)

where $n$ is the null vector to define the light-cone gauge referred to appendix B. The Jacobian in this variables has the simple form of

\[
d^n k = \frac{dx}{2x} d^{n-2} k_{\perp} dk^2.
\] (2.86)

Now let me first consider the case of unpolarized DIS. Then the required squared matrix is that of $(\sigma_{++} + \sigma_{+-})$ where ± sign shows the helicity state of virtual photon (left index) and incoming parton (right). For the calculation, the lower on-shell legs must be substituted by quark polarization, usually denoted $u(p, s)$ and $\bar{u}(p, s)$. After taking the sum of the helicity states it gives $\hat{p}$ to the lower end of the kernel $K_0$, it is expressed as $K_0(k, p)\hat{p}$. Then, it is in general decomposed into the following terms when the dimensional regularization is applied.

\[
K_0(k, p)\hat{p} = A k_{\parallel} + B k_{\perp} + C, \frac{k^2}{p n} + D \frac{k_{\perp}^2}{p n} + E \frac{k_{\perp} k_{\parallel}}{p n} \frac{1}{p n},
\] (2.87)

where the scalar factors $A\tilde{E}$ are dimensionless collinear finite functions of $A(k^2/\mu^2, x, \epsilon), \cdots$. Then the integration over $k$ giving rise to the poles in $\epsilon$
is of the form;
\[
\int_0^{q^2} d\frac{k^2}{k^2} \int_0^{-(1-x)k^2} d\frac{k_\perp^2}{k^2} \left( \frac{k_\perp^2}{\mu^2} \right)^\epsilon \left( A \left( \frac{k^2}{\mu^2}, x, \epsilon \right) k_\parallel + B \left( \frac{k^2}{\mu^2}, x, \epsilon \right) k_\perp + \cdots \right),
\]
where \( x \) integral is put aside, and the upper limit of \( k^2 \) integration is set in some high energy scale, say \( q^2 \), which play a part in other 2PI kernel. Taking into account that \( k^2_\perp \approx k^2 \) in the integration because of the on-shell condition of in this kernel, we can see that the IR divergence comes only from the term proportional to \( k_\parallel \) in the on-shell edge \( k^2 \to 0 \) by dimensional counting and it surely gives logarithmic divergence. In dimensional regularization, it appears as singularity on \( \epsilon \). To pick up only this contribution from other part of the integral, we find that a method to apply \( \not{n} \) from its left and take the trace works for this purpose because \( n^2 = n k_\perp = 0 \). The operation is expressed as \( \frac{1}{n} \not{n} \). Considering the normalization factor, it turns out that it be properly \( \frac{1}{n}/(4 \cdot n) \);
\[
K_0(k, p) \not{p} = \frac{1}{4k \cdot n} \left[ \not{n} K_0 \not{p} \right] + \text{finite part}.
\]
Regulating that the operator \( \not{k} \) means picking up \( k_\parallel = xp \) part from \( k \), then the projection operator \( \not{p} \) related quark inner line turns out to be \( \not{k} \not{n} \not{p}/(4k \cdot n) \). Although we discussed for the case that both pairs of legs are quark lines, we can get also for gluon projection operator. Even in case of gluon, the only contribution to the collinear divergent comes from the term with \( k = xp \) in \( k^2 \to 0 \) as expected. The only difference from quark case is that there is another contribution to the divergent term, proportional to \( k^2_\perp/k^2 \) which is corresponds to soft divergence from gluon. Whereas quark external lines do not provide soft divergence. \[85\].

Let me denote right half of the projection operator as \( L_{q,g} \) and left half as \( U_q \) separately following \[115\] because we have to choose appropriate \( U \) and \( L \) corresponding to the legs of the 2PI kernel. These \( U, L \) for unpolarized DIS are given in appendix \[B\]. We can directly check them in LO easily for any of leg combinations among \( q \) and \( g \). In case of longitudinally polarized kernel, first note that the required matrix squared is for \( (\sigma_{++} - \sigma_{+-}) \). Therefore \( \Delta L_{q,g} \) should be the difference of the helicity states. The derivation of \( \Delta L_{q,g} \) is straightforward through the discussion in appendix \[B\] We can easily guess that corresponding \( \Delta U_{q,g} \) should be like given in appendix \[B\] and directly check in LO as well. One thing we have to note in polarized case is that \( \gamma_5 \) and \( \epsilon^{\rho\sigma} \) is the object defined exactly in 4 dimension and there is no natural extension to \( n \) dimension needed for the dimensional regularization.
Usually the method called HVBM scheme \[116, 117\] is applied to handle this difficulty \[118\], where these quantities are defined as purely 4 dimensional object and the relation with \(n\) dimension object changes depending on 4 dimension component or \(n - 4\) dimension component. This treatment of \(\gamma_5\) is closely related to axial anomaly and gives interesting insight on polarized DIS \[119\]. We are not going into its detail here. If we are interested, please refer to those references. As a final remark those unpolarized and polarized projection operators are nearly unchanged for time-like case \[115, 120\].

Now we can see that the integration between a (2PI) kernel \(K\) and \(\mathcal{P}K\) reduces to integration on \(x\). Noting the eq. \[2.79\], then the factorized form of eq. \[2.78\] can be re-expressed as

\[
F \left( \frac{Q^2}{\mu^2}, x_B, \alpha_s, \frac{1}{\epsilon} \right) = \int_{x_B}^{1} \frac{dx}{x} C \left( \frac{Q^2}{\mu^2}, x_B, \alpha_s \right) \Gamma \left( x, \alpha_s, \frac{1}{\epsilon} \right),
\]

(2.90)

where

\[
F = [ M\not{p}] = \left[ C_0 \frac{1}{1 - K_0 \not{p}} \right],
\]

(2.91)

\[
C = \left[ C_0 \frac{1}{1 - (1 - \mathcal{P})K_0 \not{k}} \right], \quad (1 < x_B = -\frac{q^2}{2pq} > 0),
\]

(2.92)

\[
\Gamma = Z_F \left( \delta(1 - x) + x \int \frac{d^4 k}{(2\pi)^d} \delta(x - \frac{n k}{np}) \left[ \frac{\not{p} K}{4kn} \frac{1}{1 - \mathcal{P} K \not{p}} \right] \right)
\]

(2.93)

where the \([\) from the left most on both sides indicate some contraction on the indices of the virtual photon, and I also took into account of the renormalization constant of the self-energy \(Z_F^{1/2}\) for the external quark lines which is needed for renormalized cross sections. The independence of \(\Gamma\) on \(Q^2\) is consequence of IR finite of 2PI kernel \[84\]. \(Z_F\) also has IR singularity on \(\epsilon\) like

\[
Z_F = 1 + \frac{1}{\epsilon} Z_{F,1}(\alpha_s) + \frac{1}{\epsilon^2} Z_{F,2}(\alpha_s),
\]

(2.94)

so it should be included in the singular part. (For the importance of \(Z\) in IR divergent cancellation, refer to \[92\].) The factorized form in eq. \[2.93\] is usually expressed diagrammatically as fig.2.19. More generally \(\Gamma\) is expressed as

\[
\Gamma_{ij}(x, \alpha_s, \frac{1}{\epsilon}) = Z_j \left( \delta(1 - x) \delta_{ij} + x \mathcal{P} \mathcal{P} \int \frac{d^4 k}{(2\pi)^d} \delta(x - \frac{n k}{np}) U_i K \frac{1}{1 - \mathcal{P} K L_j} \right)
\]

(2.95)

where "PP" denotes the fact of extracting only pole part of the expression on its right with the projection \(\mathcal{P}\) explicitly and \(i, j\) run over \(q, \bar{q}, g\). This
Figure 2.19: diagrammatic expression of the factorization

is also true for polarized case with replacing $U, L$ by $\Delta U, \Delta L$. Noting the expansion eq. 2.80, the actual diagrams needed for two loop level (next-to-leading order, NLO) are expressed topologically as eq. 2.20. (The one loop level (leading order, LO) is a simple ladder kernel.) The integration of the

\[ f \otimes g = \int_1^1 dy f(x/y) g(y) = \int_0^1 \int_0^1 dz dy f(z) g(y) \delta(x - y). \]  

(2.96)

These convolution integral reduces to usual multiplication by taking $n$th
moment on $x$, called Mellin moment;

$$f(n) = \int_0^1 dx x^{n-1} f(x), \quad f \otimes g \rightarrow f(n)g(n) , \quad (2.97)$$

which would be greatly used in my analysis. The decomposition into the moments is obvious in DIS case. However, in general, it cannot be performed so naturally especially for angle dependent quantities. It should be noted, however, that we can still force to Mellin moment style even in those cases \[121\]. Then taking the Mellin moment on $x_B$ in both sides of eq. \[2.90\]

$$F\left(\frac{Q^2}{\mu^2}, N, \alpha_s, \frac{1}{\epsilon}\right) = C\left(\frac{Q^2}{\mu^2}, N, \alpha_s\right) \Gamma\left(N, \alpha_s, \frac{1}{\epsilon}\right) , \quad (2.98)$$

Now we observe the explicit analogy to the renormalization discussed in the previous section. $F$ can be regarded as a raw quantity which has singularity as its nature, $\Gamma$ is a renormalization constant which contains all the singularities coming from interaction, and $C$ is a renormalized quantity which is free from the divergences, table \[2.1\].

Now we are ready to see the resummation effect of the above factorization process. Let me apply the infinitesimal operation $\mathcal{D}$ of the renormalization group on $\mu$ of both sides of eq. \[2.98\] In the following let me denote $\mu$ in eq. \[2.98\] as $\mu_F$ to distinguish from the renormalization scale $\mu_R$

$$\mathcal{D} = \mu_F \frac{\partial}{\partial \mu_F} + \beta(g_s, \epsilon) \frac{\partial}{\partial g_s} , \quad (2.99)$$

where

$$\beta(g_s, \epsilon) = \mu_F \frac{\partial}{\partial \mu_F} g_s(\mu_R) , \quad (2.100)$$

in this case. Thus only in the case that $\mu_R = c \mu_F$ with $c$ arbitrary factor, the $\beta$ can be identified with that from the renormalization defined in eq. \[2.27\] and we can use same scaling property for $g_s(\mu_F)$ but still with the initial condition on $g_s(\mu_R)$. ($g_s$ in a theory should be always unique.) We are going to concentrate only on this case. Note that the left hand side of eq. \[2.98\] should not be dependent on $\mu_F$ because it is a raw (bare) quantity in sense of the factorization. Then the application of $\mathcal{D}$ gives

$$\mathcal{D} \ln F = 0 = \mathcal{D} \ln C + \mathcal{D} \ln \Gamma , \quad (2.101)$$

i.e.,

$$(\mathcal{D} - \gamma(N, \alpha_s)) C\left(\frac{Q^2}{\mu_F}, N, \alpha_s\right) = 0 , \quad (2.102)$$

47
where $\gamma(N, \alpha_s)$ is defined as
\[
\gamma(N, \alpha_s) = -\mathcal{D}\ln\Gamma = -\beta(g_s, \epsilon)\frac{\partial}{\partial g_s}\ln\Gamma\left(N, \alpha_s, \frac{1}{\epsilon}\right).
\] (2.103)

$\gamma$ does not contain poles in $\epsilon$ since both $\mathcal{D}$ and $C$ do not have. Using the expansion
\[
\Gamma\left(N, \alpha_s, \frac{1}{\epsilon}\right) = 1 + \sum_{i=1}^{\infty} \frac{1}{\epsilon^i} \Gamma_i(N, \alpha_s),
\] (2.104)
we obtain from eqs. 2.27 and 2.102 that
\[
\gamma(N, \alpha_s) = -\alpha_s \frac{\partial}{\partial \alpha_s} \Gamma_1(N, \alpha_s).
\] (2.105)

(In this step, the condition of $\mu_F = \mu_R$ becomes crucial requirement.) Note that only the coefficient of the single pole $\Gamma_1$ plays role here. For a further comparison with the OPE results, let us parametrize $\Gamma_1$ as
\[
\Gamma_1(N, \alpha_s) = -\sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k \frac{1}{k} \gamma_{k-1}(N),
\] (2.106)
so that
\[
\gamma(N, \alpha_s) = \sum_{i=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^i \gamma_{i-1}(N) = \left(\frac{\alpha_s}{4\pi}\right)^0 \gamma_0(N) + \left(\frac{\alpha_s}{4\pi}\right)^2 \gamma_1(N) + \cdots.
\] (2.107)

I will briefly show that the $\gamma(N)$ surely consistent with the anomalous dimension of the twist two Wilson operators in section 2.5. Then eq. 2.103 can be integrated and we get
\[
\Gamma\left(N, \alpha_s, \frac{1}{\epsilon}\right) = \exp\left(-\int_0^{\alpha_s} \frac{d\lambda}{\lambda} \frac{\gamma(N, \lambda)}{2\beta(\lambda) + \epsilon}\right),
\] (2.108)
where
\[
\tilde{\beta}(\alpha_s) = \frac{1}{g_s} \beta(g_s).
\] (2.109)

After all, making $\mu_F$ and $\mu_{0,F}$ ($\mu_F \ll \mu_{0,F} \sim Q^2$) dependence of $\alpha_s$ explicit, eq. 2.98 can be written as
\[
F\left(\frac{Q^2}{\mu_F^2}, N, \alpha_s, \frac{1}{\epsilon}\right) = C\left(\frac{Q^2}{\mu_{0,F}^2}, N, \alpha_s(\mu_{0,F})\right) \exp\left(-\int_0^{\alpha_s(\mu_{0,F})} \frac{d\lambda}{\lambda} \frac{\gamma(N, \lambda)}{2\beta(\lambda) + \epsilon}\right).
\] (2.110)
Now we can surely confirm the factorization of the collinear divergence appearing in each 2PI kernel connection and the resummation of large logs related to \((Q^2/\mu^2)\) through the \(\beta\) function and anomalous dimension \(\gamma\) defined as eq. 2.103. Compared with the OPE result eq. 2.50 we can see the exact correspondence if \(\gamma(N, \alpha_s)\) could be identical with the anomalous dimension \(\tilde{\gamma}^T_{O_n}(\bar{g}(t'))\) in eq. 2.50 for QCD case.

To handle the \(\epsilon\) pole in the both sides of eq. 2.110 let us introduce the “bare” density of quarks inside the hadron \(q_{B,H}(x, \alpha_s, 1/\epsilon)\). The partonic cross section eq. 2.110 should be convoluted with the density, and the hadronic cross section expected to be free of the pole. In other words, the long distance behavior in the partonic cross section should be absorbed into the unknown hadronic structure. According the LE, \(q_{B,H}\) can be expressed as

\[
q_{B,H}(x, \alpha_s, 1/\epsilon) = x \int \frac{d^mp}{(2\pi)^m} \delta \left( x - \frac{p_{Hn}}{p_{H}} \right) \left[ \frac{1}{4pn} H(p_{H} + m_{H}) \right], \tag{2.111}
\]

where \(H\) is 2PI hadron \(\rightarrow\) quark kernel and \(p_{H}\) is the hadron momentum which has a mass \(m_{H}\), fig. 2.21. For our purpose, we need one property of

\[\text{H}, \text{i.e. that it is soft in the following sense:} \]

\[
H(p^2, p_{H}^2, x) = \frac{1}{p^2} \int d^{m-2}p_{L} \left[ \frac{1}{4pn} H(p_{H} + m_{H}) \right] \lesssim \frac{c}{p^2|\delta|}, \tag{2.112}
\]

\((p_{H}^2 \ll |p^2| \rightarrow \infty, \delta > 0)\)
This assumption of softness of parton in $p_\perp$ direction in high energy limit is consistent with the observation of the impulse approximation in DIS, etc. Assuming the softness of the hadronic kernel, we can extend the $p^2$ integral to infinity;

$$q_{B,H} \left( x, \alpha_s, \frac{1}{\epsilon} \right) = \int^{-Q^2} dp^2 \frac{dp^2}{p^2} H(p^2, p_H^2, x) = \int^{-\infty} dp^2 \frac{dp^2}{p^2} H(p^2, x) + O \left( \left( \frac{p_H^2}{Q^2} \right)^\delta \right);$$

(2.113)

where in the integrand in the final form, I neglected the mass of the hadron. Therefore the “bare” density contains only power-like corrections to the $Q^2$ dependence of the cross section, which accord with higher twists in the Wilson operator expansion. In the process where those corrections are negligible because of the existence of high energy interaction scale between partons, the effect of the mass of the hadron can be neglected in the description of total interaction. On the other hand, the lower limit of the $dp^2/p^2$ integral will generate the IR singularity which must be exactly cancel with the divergences from partonic cross section. This is because the hadron state can be considered as a bound state in QCD merging all the degenerate stats in sense of perturbation and the degenerate state process would absorb the IR divergences in the partonic process, which is nothing but Kinoshita-Lee-Nauenberg theorem explained in the previous section. Note that this kind of assumption also appears in the application of OPE to DIS. The proof of OPE is done between partonic states. For the application to DIS, we assume the insensitivity of OPE structure on the hadron state.

Therefore, with the “bare” density, we can define “dressed” density, called parton distribution function, which is free from the collinear singularity;

$$q_H(N, \mu^2_{0,F}) = \exp \left( - \int_0^{\alpha_s(\mu_{0,F})} d\lambda \frac{\gamma(N, \lambda)}{\lambda 2\beta(\lambda) + \epsilon} \right) q_{B,H} \left( N, \alpha_s, \frac{1}{\epsilon} \right).$$

(2.114)

Then the density $q_H$ develops with $\mu^2_{0,F}$ according to

$$\mu^2_{0,F} \frac{\partial}{\partial \mu^2_{0,F}} q_H(N, \mu^2_{0,F}) = - \frac{1}{2} \gamma(N, \alpha_s(\mu_{0,F})) q_H(N, \mu^2_{0,F}).$$

(2.115)

When we define $P_{qq}(x, \alpha_s)$ as

$$\int_0^1 dx x^{N-1} P_{qq}(x, \alpha_s) = - \frac{1}{2} \gamma(N, \alpha_s)$$

(2.116)

we can rewrite eq. (2.115) as

$$\mu^2_{0,F} \frac{\partial}{\partial \mu^2_{0,F}} q_H(x, \mu^2_{0,F}) = \int_x^1 \frac{dz}{z} P_{qq}(\frac{x}{z}, \alpha_s(\mu_{0,F})) q_H(z, \mu^2_{0,F}).$$

(2.117)
When the cut is applied, this is nothing but Altarelli Parisi equation [40], now called DGLAP equation. In the following the application of cutting rule is premised for its physical meaning. Let us expanding \( P_{qq} \), called splitting function, in powers of \( \alpha_s \):

\[
P_{qq}(x, \alpha_s) = \left( \frac{\alpha}{4\pi} \right) P_{qq}^{(0)}(x) + \left( \frac{\alpha}{4\pi} \right)^2 P_{qq}^{(1)}(x) + \cdots.
\]  

(2.118)

Then, from eqs. 2.95 and 2.104, we can obtain the function \( P_{qq} \) in our framework as

\[
\Gamma_{qq}\left(x, \alpha_s, \frac{1}{\epsilon}\right) = \delta(1-x) + \frac{2}{\epsilon} \left( \left( \frac{\alpha}{4\pi} \right) P_{qq}^{(0)}(x) + \left( \frac{\alpha}{4\pi} \right)^2 P_{qq}^{(1)}(x) + \cdots \right) + O\left( \frac{1}{\epsilon} \right).
\]  

(2.119)

From eq. 2.95 we can exhaust the other splitting functions \( P_{i,j} \) where \( i, j \) run over \( q, \bar{q}, g \) as well. Then the eq. 2.117 is extended to

\[
\mu_0^2 \frac{\partial}{\partial \mu_0^2} q_{i,H}(x, \mu_0^2, N, \alpha_s(\mu_0^2)) = P_{ij}(x, \alpha_s(\mu_0^2)) \otimes q_{j,H}(x, \mu_0^2, N, \alpha_s(\mu_0^2)),
\]  

(2.120)

where the symbol \( \otimes \), eq. 2.96, is applied, \( i, j \) runs over \( q, \bar{q}, g \) and the summation over \( j \) is understood.

After all, with the convolution with \( q_{H}(N, \alpha_s(\mu_0^2)) \) in eq. 2.5 the partonic cross section \( F \), eq. 2.110 can be written as hadronic one \( F^H \) which has no collinear singularity.

\[
F^H\left( \frac{Q^2}{\mu_F^2}, N, \alpha_s \right) = C\left( \frac{Q^2}{\mu_0^2}, N, \alpha_s(\mu_0^2) \right) q_{H}(N, \mu_0^2).
\]  

(2.121)

This completes the factorization, i.e. \( C \) contains all the short distance scale physics and \( q_{H} \) absorbs all the unknown long distance physics in the hadronization as the initial conditions of DGLAP evolution. The scale transformation is controlled by the running coupling and the splitting functions. This diagrammatic way of the factorization can be generalized to time-like process and much complex processes including proton-proton process unlike OPE. As it clear from its derivation, the scale property of \( q_{H} \) is independent on the short distance information, i.e. that what kind of hard interaction the partons are included into. Therefore \( q_{H} \) are general fundamental language to study high energy interactions together with hadrons.

### 2.4 Distribution Functions

As given in the previous section, the parton distribution functions (PDFs) contains all the information on the inner structure of hadron, and the extrac-
tion of those in the framework of perturbative QCD is the central issue of this thesis. Therefore I would like to look closer at the property of those distributions. The bare PDFs are diagrammatically expressed as fig. 2.22. We can also define time-like distribution functions, called fragmentation functions (FFs) or decay functions, indicated pictorially as fig. 2.22. The definition of

\[
\text{PDF: space-like} \quad \text{FF: time-like}
\]

Figure 2.22: parton distribution function and fragmentation function: \( p_H \) in the short interaction part is treated as massless as mentioned in the end of the previous section.

the PDFs is to some extent a matter of convention. The most commonly used convention is the \( \overline{\text{MS}} \) definition \[122, 123\] based on the above diagrammatic interpretation, which arose originally from OPE method \[75\]. Using the \( \overline{\text{MS}} \) definition, we can formulate the diagrammatic expression of PDFs on more field theoretical basis. In the following I’m going to introduce distributions in our current interest, unpolarized PDFs \[122, 124\], longitudinally polarized PDFs (parton helicity distributions) \[119, 125\] and unpolarized FFs (decay functions) \[122, 124\]. As discussed in the previous section, all of these bare distributions receive the radiative corrections, and evolves as the scale \( \mu_0^2 \) changes. DGLAP evolutions with the appropriate splitting functions control those behavior.

\section*{2.4.1 Unpolarized PDF}

The unpolarized quark distribution in a hadron is given as the hadron matrix element of certain quark field operators with the null place coordinates with

\[52\]
coordinates defined by $n$ in the previous section. Then we can define $k^+$ direction by $n \cdot k$. The bare density eq. 2.111 can be written as a matrix element of bi-local operator with the quark (formal asymptotically) free field operator $\psi_i$ for flavor $i$ as

$$q_{i,B,H}(x, \alpha_s, \epsilon) = \int \frac{d^m k}{(2\pi)^m 4k^+} \frac{d^m y \delta \left( x - \frac{k^+}{p_H^+} \right)}{m^4} e^{-ik^+y} \langle p_H | \bar{\psi}_i(y) \gamma^+ \psi_i(0) | p_H \rangle. \quad (2.122)$$

Here $|p_H\rangle$ represents the state of a hadron with momentum $p$ aligned so that $p_T = 0$, $1/\epsilon$ is abbreviated as $\epsilon$ and color sum is presumed. (Note that we are using the light-cone gauge.) After some algebra for the light-cone coordinates, we can obtain

$$q_{i,B,H}(x, \alpha_s, \epsilon) = \frac{1}{2} \int \frac{dy^-}{2\pi} e^{-ixp^-_H y^-} \langle p_H | \bar{\psi}_i(0, y^-, 0) \gamma^+ \hat{F} \psi_i(0) | p_H \rangle, \quad (2.123)$$

where I put artificially the gauge link $F$ which makes the operator gauge invariant and will be explained below. The quark field operator $\psi_i(0)$ evaluated at $x = 0$, and annihilates a quark in the hadron at that point. While the operator $\bar{\psi}_i(0, y^-, 0)$ recreates the quark at $x^+ = x_T = 0$ and $x^- = y^-$, where we take the appropriate Fourier transform in $y^-$ so that the quark which was annihilated and recreated has momentum $k^+ = xp^+_H$. Thus the insertion of the projection operator which is proportional to $\gamma^+$ corresponds to an extraction of parton state with light-cone configuration defined by $\gamma^+$. The motivation for the definition is that this is the hadron matrix element of the appropriate number operator for finding a quark $i$ as shown later. There is still one subtle point. The number operator idea corresponds to a particular gauge, light-cone gauge $A^+ = 0$. If we are using any other gauge, we insert the (Wilson line [21]) operator

$$\hat{F} = P \exp \left( -ig \int_0^y dz^- A^+_a(0, z^-, 0) t_a \right). \quad (2.124)$$

The $P$ indicates a path ordering of the operators and color matrices along the path from $(0,0,0)$ to $(0, z^-, 0)$. This operator is called gauge link, which is introduced to make the bilocal operator gauge invariant, and in the light-cone gauge reduces to unity.

Following the same argument as above, with appropriate insertion oper-
ators we can also define anti-quark and gluon distributions as

\[ q_{i,B,H}(x, \alpha_s, \xi) = \frac{1}{2} \int \frac{dy}{2\pi} e^{-ixp_H y^-} \langle p_H | \text{Tr} \{ \gamma^+ \psi_i(0, y^-) \} \bar{\psi}(0) \} | p_H \rangle, \tag{2.125} \]

\[ g_{B,H}(x, \alpha_s, \xi) = \frac{p_H^+}{x} \int \frac{dy}{2\pi} e^{-ixp_H y^-} \langle p_H | A^\mu_a(0, y^-) A_{a,\mu}(0) | p_H \rangle. \tag{2.126} \]

Note that those normalization factors are defined as in the free field theory those distribution satisfy \( q(g)(x) = \delta(1 - x) \). To convert the gluon distribution into gauge invariant form (the anti-quark distribution is obvious), we should note that in the light-cone gauge \( n^\cdot A = A^+ = 0 \). Then \( F^{+\mu} \) simply becomes \( F^{+\mu} = \partial^+ A^\mu \) with the solution

\[ A^\mu(\xi) = \frac{1}{2} \int d\eta F^{+\mu}(\xi^+, \xi^-) \theta(\xi^- - \eta) + \text{const.}, \tag{2.127} \]

where the constant is determined by the boundary condition, but is not important here. Then the gluon distribution is written as

\[ g_{B,H}(x, \alpha_s, \xi) = \frac{1}{xp_H^+} \int \frac{dy}{2\pi} e^{-ixp_H y^-} \langle p_H | F^{+\mu}_a(0, y^-) | p_H \rangle, \tag{2.128} \]

where I inserted \( \bar{F} \) to obtain totally gauge invariant distribution.

As described in [122,123], the PDFs of eqs. 2.123, 2.125 and 2.128 can be identified with the twist two operators \( O^N \) eqs. 2.10 and 2.11 appearing in the OPE after taking appropriate sum of the \( n \)th moments \( \int_0^1 dx x^{N-1} q(x) \) of those distributions. Note that the coefficients of the Taylor expansion on \( y^- \) variable gives the similar structure as \( O^N \). (As will be briefly shown in the next section, the anomalous dimensions of \( O^N \) can be also identified with \( \gamma(N) \) eq. 2.105. Therefore the two methods are surely gives the same description on the DIS process.) These distributions are know as twist two distributions, which dominate the the interactions in high energy domain. The following distributions, helicity distributions and fragmentation functions are also twist two distributions.

The physics of the definition of eq. 2.123 in DIS process in the parton model (no IR radiative correction) description is illustrated in fig. 2.23. Let us assume the virtual photon momentum \( q \) lies on the \( p_H^0 = p_H^3 \) plane, i.e. \( q_T = 0 \). Then consider the short distance interaction with no radiative correction between the virtual photon and the massless quark with momentum \( k^+ = xp_H^+ \) (with other components zeros), where \( p_H^2 \ll q^2 = Q^2 \) and \( p_H^+ \) dominate.
Figure 2.23: illustration of DIS in null plane coordinate: the red circle in the right figure has the localized size of $|\Delta x| \sim 1/Q$.

frame is chosen. Note that the consistency with the momentum configuration of the Sudakov decomposition eq. 2.81 with $k_\perp = 0$. Then the hadron state can be described as as the band which shrinks in $x^-$ direction and expands into $x^+$ direction, fig 2.23. Then the interaction of constituent quark with the virtual photon $q = (q^+, -Q^2/2q^+)$ strikes the quark and makes it on-shell. We see the struck quark aligns in $x^-$ plane ($xp_H^+ + q^+ = 0$). The photon interaction is localized to within $|\Delta x| \sim 1/Q$. During this short time interval, the quarks and gluons in the hadron are effectively free, since their typical interaction times are considered to be comparatively much longer. (This is corresponding to the assumption of “softness” in the previous section.)

2.4.2 Longitudinally Polarized PDF (Parton Helicity Distribution)

As will be introduced in the following chapter, longitudinally polarized DIS, DIS with longitudinally polarized beam (virtual photon) and target, is the process to access to the helicity structure of the target hadron. The process is described by (longitudinally) polarized PDF $\Delta q(g)$ defined by the insertion of $\Delta U$ to longitudinally polarized hadron state (helicity eigen state), fig. 2.24. The polarized PDF is also called parton helicity distribution to make its property clear. As given in appendix A, the helicity state is given as $|P, \sigma\rangle$ with the definition of the covariant polarization vector $W^\mu$, Pauli-Lubanski vector. (For nucleon case $\sigma = \pm 1/2$) We can say that the longitudinally polarized DIS, simply called polarized DIS in the following, is the process we can...
access to the difference of the helicity states of parton in polarized nucleon. Then the bare quark polarized distribution is written like the unpolarized case as

$$
\Delta q_{i,B,H}(x, \alpha_s, \frac{p_H}{2}) = \frac{1}{2} \int \frac{dy}{2\pi} e^{-ixp_H y} \langle p_H, \sigma | \bar{\psi}_i(0, y^-) \gamma^+ \gamma_5 F \psi_i(0) | p_H, \sigma \rangle .
$$

(2.129)

(We can confirm from parity invariance that $\Delta q_{\sigma} = -\Delta q_{-\sigma}$.) Following the discussion above and the projection operator in the appendix [13] we see that the gluon helicity distribution is defined as

$$
\Delta g_{B,H}(x, \alpha_s, e) = \frac{1}{xp_H^+} \int \frac{dy}{2\pi} e^{-ixp_H^+ y^-} \langle p_H, \sigma | F_+^{a\mu}(0, y^-) \hat{F} \hat{F}_{a\mu+}(0) | p_H, \sigma \rangle ,
$$

(2.130)

where $\hat{F}$ is the dual field tensor

$$
\hat{F}_a^{\mu\nu} = \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{a,\rho\sigma} .
$$

(2.131)

The anti-quark helicity distribution is obvious from eq. 2.129.

When we define $P_\pm$ and $\psi_\pm$ as

$$
P_\pm = \frac{1}{2} \gamma^+ \gamma^\pm \quad (P_\pm^2 = P_\pm) ,
$$

(2.132)

$$
\psi = \psi_+ + \psi_- \quad \psi_\pm = P_\pm \psi ,
$$

(2.133)

in $p_H \rightarrow \infty$, $\psi_+$ dominates over $\psi_-$, which serves as higher twists in the matrix elements and is neglected below. Using the relations

$$
\bar{\psi} \gamma^+ \psi = \sqrt{2} \psi^1 \psi ,
$$

(2.134)

$$
\bar{\psi} \gamma^+ \gamma_5 \psi = \sqrt{2} \psi^1 \gamma_5 \psi ,
$$

(2.135)

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then we can rewrite eqs. 2.123 and 2.129 with flavor index omitted as

$$q_H(x) = q^+(x) + q^-(x),$$  
$$\Delta q_H(x) = q^+(x) - q^-(x),$$

where

$$q_\pm(x) \equiv \frac{1}{\sqrt{2}p_\pm^H} \int \Pi_X \sum_X \delta((1 - x)p_\mp^H - P_X) \left| \langle X \left| \frac{1 \pm \gamma_5}{2} \psi_+(0)|p_H, \sigma \rangle \right|^2. \quad (2.138)$$

This is pictorially shown in fig. 2.25. The reformulation shows manifestly

\[ \triangleleft \text{Figure 2.25: PDFs with } q_+ \text{ and } q_- \\triangleleft \]

the probabilistic nature of $q_\pm$, i.e. $q_\pm$ gives the probability of finding inside (extracting from) the hadron a quark with longitudinal momentum $xp_\pm^H$ and helicity $\pm$ states. From that conclusion, we obtain the bounds for $\Delta q(x)$:

$$|\Delta q(x)| \leq q(x), \quad (2.139)$$

which is called positivity condition. For anti-quarks and gluons, we can obtain the similar conclusions. For the other parton distributions exhausting all the possible distributions, refer to [125, 119].

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2.4.3 Fragmentation Function

I’m going to mention only unpolarized fragmentation functions (FFs) for our current interest. The fragmentation functions defined from the diagram fig. 2.22. The quark and antiquark FF are defined as

\[
D_{q,B}^H(z, \alpha_s, \beta)(z, \alpha_s, \beta) = \int \frac{dy^-}{12\pi} e^{-ip_H y^-/z} \sum_X \int d\Pi_X \text{Tr} \{ \langle 0 | q^+(0, y^-, 0) \tilde{F} p_H X \langle p_H X | \tilde{F}' \tilde{\psi}_i(0) | 0 \rangle \},
\]

(2.140)

and gluon fragmentation function as

\[
D_{g,B}^H(z, \alpha_s, \beta) = \int \frac{dy^-}{12\pi} e^{-ip_H y^-/z} \sum_X \int d\Pi_X \langle 0 | \tilde{\psi}_i(0, y^-, 0) \tilde{F} p_H X \langle p_H X | \tilde{F}' q^+(0) | 0 \rangle \},
\]

(2.141)

and (2.142)

and gluon fragmentation function as

\[
D_{g,B}^H(z, \alpha_s, \beta) = \int \frac{dy^-}{16\pi} e^{-ip_H y^-/z} \sum_X \int d\Pi_X \langle 0 | F_a^{+\mu}(0, y^-, 0) \tilde{F} p_H X \langle p_H X | \tilde{F}' F_{a, \mu+}(0) | 0 \rangle \}.
\]

(2.143)

Here \( \tilde{F}' \) and \( \tilde{F} \) are the path integrals similar to eq. 2.124. As we can guess from fig. 2.22, in the light-cone gauge FFs have interpretation as the probability (number) density for an original parton \( q(g) \) to produce a hadron \( H \) with fraction \( z \) of its momentum. Note that the fraction \( z \) appeared in the denominator of the Fourier power. Unlike parton distribution functions, these functions do not relate to the local operators because the observed hadron comes in the final state and the matrix element can not be described as a unit bilocal operator. The OPE expression for PDFs is the basis for first principle computations of some of their integer moments within lattice QCD [35, 36]. Thus similar calculations cannot be pursued for fragmentation functions although there is the great similarity in the diagrammatic shape. Some studies based on the diagrammatic similarity can be found in [127, 84, 128].

2.5 Consistency with OPE Result

As the last section of this chapter, I’m going to briefly show the consistency of the anomalous dimension \( \gamma \) in eq. 2.105 with the anomalous dimension
given from OPE method in dimensional regularization and $\overline{MS}$ scheme. In the following, I’m going to concentrate only on the non-singlet anomalous dimension.

In OPE, the anomalous dimension is obtained from the renormalization of the radiative corrections to the operator

$$O^N_{\mu_1 \cdots \mu_N}(x) = \{ \bar{q}(x) \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_N} q(x) \} s, t$$ \hspace{1cm} (2.143)

where $S, T$ denote the symmetric and traceless part, like eq. 2.10. The lowest twist contribution can be extracted by considering the two point quark Green functions with $O^N$ insertion,

$$\frac{1}{4} \langle p | O^N_{\mu_1 \cdots \mu_N} | p \rangle = \int d^4 x_2 d^4 x_1 e^{i p (x_1 - x_2)} \langle 0 | T[O^N_{\mu_1 \cdots \mu_N}(0) \psi(x_1) \bar{\psi}(x_2)] | 0 \rangle = O^N(\alpha_s, \epsilon) p_{\mu_1} \cdots p_{\mu_N} - \text{traces}$$ \hspace{1cm} (2.144)

and by taking only the first term. The factor $1/4$ normalizes $O^N$ to unity in the free field limit. $O(\alpha_s, \epsilon)$ denotes the renormalized matrix element, related to the bare one in $D = 4 - \epsilon$ dimension as follows

$$O^N_B(\alpha_s, 0, \frac{\mu_R^2}{p^2}, \epsilon) = Z_N^{-1}(\alpha_s(\mu_R), \frac{1}{\epsilon}) \ O^N(\alpha_s(\mu_R), \epsilon, \frac{p^2}{\mu_R^2})$$ \hspace{1cm} (2.145)

Here I showed $\mu_R$ dependence in the bare operator through QCD coupling in dimensional regularization explicitly with bare coupling $\alpha_{s,0}$, and $\mu_R$ dependence of the renormalized coupling $\alpha_s$ is also explicitly indicated. The collinear divergences are regularized by taking $p^2 \neq 0$. The anomalous dimension $\gamma^N(\alpha_s)$ is defined as

$$\gamma^N(\alpha_s) = -\mu_R \frac{\partial}{\partial \mu_R} \ln Z_N(\alpha_s(\mu_R), \frac{1}{\epsilon})$$ \hspace{1cm} (2.146)

Noting that $Z_N$ depends on $\mu_R$ only through $\alpha_s$, the argument from eq. 2.103 to eq. 2.107 is applicable replacing $\Gamma$ by $Z_N$ with the help of eq. 2.100. Therefore, the problem now reduces to find a relation between $Z_N$ in OPE and $\Gamma$ in mass factorization.

The $O^N$ part in eq. 2.144 can be projected out by contracting the Lorentz indices with an arbitrary light-like vector $m_\mu$ ($m^2 = 0$):

$$O^N(\alpha_s, \epsilon) = \langle p | \{ \bar{q} \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_N} q \} s, t | p \rangle \frac{m_{\mu_1} \cdots m_{\mu_N}}{(m \cdot p)^N}$$ \hspace{1cm} (2.147)
This works owing to the fact that trace terms which are proportional to $g^\mu\nu$ drops by the null vector. As $O^{N}_{\mu_{1}\ldots\mu_{N}}$ is a gauge-invariant operator, its renormalization constant $Z_{N}$ can be calculated in any gauge in general as far as $\mathcal{MS}$ is used (gauge invariant method). Let us therefore consider eq. 2.147 in the light-cone gauge and take $m = n$. Since $n^\mu A_\mu = 0$, the covariant derivatives are reduced to usual derivative $\partial_\mu$, i.e. there are no effective vertices with gluon insertions. Whereas for the simple quark-quark vertex we realize it can be written as $\sqrt{n} (k \cdot n)^{N-1}$, where $k$ is the momentum entering the vertex with $O^N$ insertion, fig. 2.26. When we take $\sqrt{n}$ for the external quark legs, eq. 2.147 can be rewritten as follows:

$$O^N(\alpha_s, \epsilon) = \int_{-1}^{1} dx \, x^{N-1} O(x, \alpha_s, \epsilon), \quad (2.148)$$

where

$$O(x, \alpha_s, \epsilon) = Z_F \left( \delta(1-x) + x \int \frac{d^d k}{(2\pi)^d} \delta \left( x - \frac{n k}{n p} \right) \left| \frac{\sqrt{n}}{4k n} T(k, p) \right| \right), \quad (2.149)$$

and $T(k, p)$ is the fully connected four-point function. Here the variable $x$ is the Feynman parameter in $k$ loop integration (negative $x$ is for anti-quark line). We can easily check plausibility of eq. 2.149 in the one loop level calculation. Thus the effective vertex for $O^N$ insertion in the light-cone gauge coincides with the $\sqrt{n}/4k n$ vertex in this method. Let us consider the left-hand side of eq. 2.145 and continue the quantity $O_B^N$ analytically to $D = 4 + \epsilon$ dimensions. Then we see that now a $p^2 \to 0$ limit does exist and is equivalent to $\alpha_s$ limit. In that case $O_B^N$ reduces to unity. Hence the renormalized $O^N$ is simply reduced to $Z_N$ in $p^2 \to 0$ limit. (Note that this

Figure 2.26: general diagram for the anomalous dimension with $O^N$ insertion in the light cone gauge: $T(k, p)$ is the fully connected four-point function.
process of the renormalization and factorization via analytic continuation of the dimension is the standard method to calculate some quantity in the mass factorization suggested in [84], and this simple analytic continuation implicitly introduces the condition $\mu_R = \mu_F$.

$$Z_N(\alpha_s, \frac{1}{\epsilon}) = \int_{-1}^{1} dx x^{N-1} O(x, \alpha_s, \epsilon) ,$$

(2.150)

where $O$ is eq. 2.149 but with $p^2 \to 0$. Now we will see the similarity of $O$ to $\Gamma$ in eq. 2.93. I’m not going into its detail, but its actually related to $\Gamma$ with the help of dispersion relation. The relation is given as follows:

$$Z_N(\alpha_s, \frac{1}{\epsilon}) = \int_{-1}^{1} dx x^{N-1} \left( \Gamma_{qq} \left( x, \alpha_s, \frac{1}{\epsilon} \right) \theta(x) - \Gamma_{q\bar{q}} \left( -x, \alpha_s, \frac{1}{\epsilon} \right) \theta(-x) \right) .$$

(2.151)

where the cutting rule is premised for $\Gamma$ and $\theta$ is the step function. For the concrete derivation related to fermion number conservation, please refer to [84]. Then comparing the coefficients of single poles on both sides of eq. 2.151, we obtain

$$-\frac{1}{2} \gamma^N(\alpha_s) = \int_{-1}^{1} dx x^{N-1} \left( P_{qq}(x, \alpha_s) \theta(x) - P_{q\bar{q}}(-x, \alpha_s) \theta(-x) \right)$$

(2.152)

$$= P_{qq}(N, \alpha_s) + (-1)^N P_{q\bar{q}}(N, \alpha_s) .$$

(2.153)

Here $(-1)^N$ factor appeared.

Thus in OPE method, we cannot access to the full information on the arbitrary moment of the splitting function. This is the same observation that the contribution of the moments to the scaling property of cross section is limited in some of the moments depending on the process, e.g. for DIS case, only even number of moments contribute to the structure function $F_2$. The parton model derivation of the splitting function given in this chapter surely provides the same results for the anomalous dimension of OPE with proper combination, but it makes no restriction on the value of $N$. In other words, the parton formula, besides reproducing the OPE result, also provides the analytic continuation of the latter to those values of $N$ which are artificially forbidden in the OPE method. This analytic property to arbitrary $N$ plays central role in the pQCD analysis with (inverse) Mellin transformation given in chapter 4.

Because the anomalous dimension regulates the scaling behavior of the operator $O^N$, this consistency of the anomalous dimension with the parton
model derivation allows us to make an identification of $O^N$ defined in eq. 2.147 as an appropriate moment of the (normalized) parton distributions defined in the axial gauge eq. . The above discussion suggests that $O^N$ with $N$ the odd number corresponds to $q(N) - \bar{q}(N)$, while $O^N$ with the even number is equivalent to $q(N) + \bar{q}(N)$. In the (longitudinally) polarized case, the additional $(-1)$ factor emerges to $q \to \bar{q}$ splitting, i.e., $\Gamma_{q\bar{q}}$, because of the projection operator $\Delta U_{\bar{q}}$ of eq. B.26 which reflects the helicity flip between quark and antiquark. Thus, in polarized case, eq. 2.153 becomes

$$-\frac{1}{2}\Delta \gamma^N(\alpha_s) = \Delta P_{qq}(N, \alpha_s) + (-1)^{N+1}\Delta P_{q\bar{q}}(N, \alpha_s). \quad (2.154)$$

Therefore the local quark operator appearing in the OPE description of the polarized DIS has an interpretation that its odd number corresponds to $\Delta q(N) + \Delta \bar{q}(N)$, and even number to $\Delta q(N) - \Delta \bar{q}(N)$. (The operator is defined as the above unpolarized case following the operator defined below, eq. 3.31 (though itself is actually singlet operator).

$$P_{qq}(N, \alpha_s) + (-1)^N P_{q\bar{q}}(N, \alpha_s) \quad \begin{cases} 
\text{N odd: for the evolution of } q(N) - \bar{q}(N) \\
\text{N even: for the evolution of } q(N) + \bar{q}(N) 
\end{cases} \quad (2.155)$$

$$\Delta P_{qq}(N, \alpha_s) + (-1)^{N+1}\Delta P_{q\bar{q}}(N, \alpha_s) \quad \begin{cases} 
\text{N odd: for } \Delta q(N) + \Delta \bar{q}(N) \\
\text{N even: for } \Delta q(N) - \Delta \bar{q}(N) 
\end{cases} \quad (2.156)$$

Thus far, we treated quark operators only assuming the their non-single combination. The gluon operator can be interpreted as the moment of gluon distribution as well, considering the equivalence of singlet anomalous dimensions in OPE to those defined in the parton model description.
Chapter 3

Application to High Energy Processes

In this chapter I’m going to list the ingredients needed for our perturbative QCD (pQCD) analysis. First I’m going to show the structure of DGLAP evolution equations which are process independent ingredient explicitly enough for its actual application. After that, I’m going to introduce the high energy processes related to my analysis, all of which can be described by the parton picture based on the factorization given in the previous chapter.

3.1 DGLAP Equations for its Application

In the demonstration of mass factorization, we concentrated in principle on 2PI ladders connected only by quark lines to make the discussion simple. There can be anti-quark and gluon lines at the same time. Even for those cases, as I made some comment on that, we can apply appropriate projection operators $U, L$ to eliminate the collinear singularities from those lines as well. Indeed the formulation of the DGLAP evolutions including them is straightforward, it seems not to adequate for the actual application for perturbative QCD (pQCD) because the discussion was somewhat abstract. Since the DGLAP evolutions are the process independent fundamental ingredients, thus, in this section, I would like to clearly formulate the DGLAP equations for quarks, anti-quarks and gluons prospecting its actual application to pQCD analysis.

Because QCD interaction has no dependence on the sort of flavor $i$ except the active flavor number $n_f$, the splitting functions are common for all the flavors. (From the property of mass independent normalization, they are
independent also on the mass of quarks. Whereas short distance cross section $C$ must be modified in case the mass could not be neglected.) Therefore we can decompose the DGLAP evolution system eq. 2.120 into the following form. In the following I’m going to omit 0 index in $\mu_F$ and $H$ in parton distributions.

Let me first redefine $P_{ij}$ in eq. 2.120 as flavor independent function. $P_{gg}$ is unchanged, and $P_{q\bar{q}}, P_{qg}$ can be redefined in a flavor independent way as

$$P_{q\bar{q}} \equiv P_{q\bar{q}} = P_{\bar{q}q}, \quad P_{qg} \equiv 2n_f P_{qg} = 2n_f P_{\bar{q}g},$$

(3.1)

where $i$ runs active flavors. Then note that any difference $q_i - q_j$ and $q_i - \bar{q}_j$ of quark and (anti-)quark distributions decouples from the gluon distribution $q_g = g$. Hence the combination maximally coupling to $g$ is the flavor-singlet distribution $q_s$

$$q_s = \sum_{i=1}^{n_f} (q_i + \bar{q}_i).$$

(3.2)

Then the evolution between $q_s$ and $g$, called singlet evolution, becomes

$$\frac{\partial}{\partial \ln \mu_F^2} \left( q_s \right) = \left( P_{qq} \quad P_{q\bar{q}} \right) \otimes \left( q_s \right).$$

(3.3)

where $P_{qq}$ is specified below. In order to decouple the evolution of the difference combinations, we make use of the general structure of the (anti-)quark - (anti-)quark splitting functions,

$$P_{q_i q_j} = P_{\bar{q}_i \bar{q}_j} = \delta_{ij} P^{V}_{qq} + P^{S}_{qq}, \quad P_{q_i \bar{q}_j} = P_{\bar{q}_i q_j} = \delta_{ij} P^{V}_{q\bar{q}} + P^{S}_{q\bar{q}},$$

(3.4)

where $P^S$ is purely flavor independent part which includes gluon ladder inside and makes flavor change possible and $P^V$ is a part which contains no gluon ladder and is diagonal for flavors, fig. 3.3. These splitting functions lead to three independent difference type evolution, called non-singlet evolutions. They are for the combinations of

$$q_{NS,ij}^\pm = q_i \pm \bar{q}_i - (q_j \pm \bar{q}_j), \quad q_{NS}^V = \sum_{i=1}^{n_f} (q_i - \bar{q}_i).$$

(3.5)

These combinations evolve with

$$P_{NS}^\pm = P_{qq}^V \pm P_{q\bar{q}}^V,$$

$$P_{NS}^V = P_{qq}^V - P_{q\bar{q}}^V + n_f (P_{qq}^S - P_{q\bar{q}}^S) \equiv P_{NS}^- + P_{NS}^S$$

(3.6)

(3.7)
Figure 3.1: ladder structure for $P_{qq}^V$ and $P_{qq}^S$ splitting functions

$$
\frac{\partial}{\partial \ln \mu_F^2} q_{NS} = P_{NS} \otimes q_{NS}.
$$  (3.8)

Finally the singlet splitting function $P_{qq}$ in eq. 3.3 is given as

$$
P_{NS}^V = P_{qq}^V + P_{q\bar{q}}^V + n_f(P_{qq}^S + P_{q\bar{q}}^S) \equiv P_{NS}^+ + P_{PS}^S.
$$  (3.9)

In the expansion in powers of $\alpha_s$, the flavor-diagonal quantity $P_{qq}^V$ in eq. 3.4 starts at the first order. $P_{qq}^V$ and the flavor-independent gluon induced contributions $P_{q\bar{q}}$ and $P_{q\bar{q}}$, i.e. $P_{PS}^S$, are of order $\alpha_s^2$. A non-vanishing $P_{NS}^S \propto P_{qq}^S - P_{q\bar{q}}^S$ occurs for the first time at the third order. These general structure are independent of not only either polarized or unpolarized but also either space-like or time-like.

Our analysis treats up to $\alpha_s^2$, i.e., NLO. Up to this order, the complete set of unpolarized space-like splitting functions can be found in [120, 115], time-like functions are in [130, 131]. The polarized space-like functions are in [132, 133], time-like functions are in [120]. A interesting discussion on the supersymmetry as a check of those results can be found in [133, 120, 134, 135, 136, 137].
3.2 Deep Inelastic Scattering and Structure Functions

In this section and the following sections, I’m going to introduce the kinematics of the high energy processes related to my analysis and the quantities described by distribution functions introduced in the previous chapter.

Deep inelastic scattering (DIS) is the typical high energy process to enable us to access the inner structure of a target hadron with the probe of lepton beam, fig 3.2.

\[
e(k, \lambda) + H(p, \sigma) \rightarrow e'(k', \lambda') + X. \tag{3.10}
\]

The lepton \(e\) is scattered hard off the target \(H\) to \(e'\). The hard scattering breaks the target and creates a large number of particles \(X\) as remnants.

Our interest is in the process where the helicity state of the final lepton state is not measured. So we consider only the processes summed over the final state lepton helicity. I do not consider weak boson contribution either here. Then the cross section of the process can be described as

\[
\sigma(ep \rightarrow e'X) = \frac{2\pi e^4}{s} \int \frac{d^3 k'}{(2\pi)^3 2k_0'} \frac{1}{q^2} L_{\mu\nu}(k, k') W^{\mu\nu}(p\sigma; q), \tag{3.11}
\]

where the lepton sector \(L_{\mu\nu}\) is given as

\[
L_{\mu\nu}(k, k') = L^{(S)}_{\mu\nu}(k, k') + iL^{(A)}_{\mu\nu}(k, k'), \tag{3.12}
\]

\[
L^{(S)}_{\mu\nu}(k, k') = k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k \cdot k, \tag{3.13}
\]

\[
L^{(A)}_{\mu\nu}(k, k') = \epsilon_{\mu\nu\rho\sigma} s^\rho (k - k')^\sigma. \tag{3.14}
\]

Figure 3.2: kinematics of DIS process
whereas the hadron sector $W_{\mu \nu}$ is as

$$W_{\mu \nu}(p \sigma; q) = W^{(S)}_{\mu \nu}(p; q) + iW^{(A)}_{\mu \nu}(p \sigma; q) ,$$  \hspace{1cm} (3.15)

$$W^{(S)}_{\mu \nu}(p; q) = \left(-g_{\mu \nu} + \frac{q_{\mu} q_{\nu}}{q^2}\right) F_1(p, q) + \frac{1}{p \cdot q} \left(p_{\mu} - \frac{p \cdot q}{q^2} q_{\mu}\right) \left(p_{\nu} - \frac{p \cdot q}{q^2} q_{\nu}\right) F_2(p, q) ,$$  \hspace{1cm} (3.16)

$$W^{(A)}_{\mu \nu}(p \sigma; q) = \epsilon_{\mu \nu \rho \sigma} \frac{q^\rho}{p \cdot q} \left(S^\sigma \eta_1(p, q) + \left[S^\sigma - \frac{S \cdot q}{p \cdot q} p^\sigma\right] g_2(p, q)\right) ,$$  \hspace{1cm} (3.17)

where $M$ is the mass of the hadron, $s^\mu$ and $S^\mu$ are the covariant spin axial vectors, as defined in appendix A of the initial lepton and target hadron which have the eigenvalues of $\lambda$ and $\sigma$ for each particle state. The mass of lepton was neglected. Note that the superscript of $(S)$ and $(A)$ on $W$ and $L$ indicates the quantity symmetric or anti-symmetric under interchange of $\mu$ and $\nu$. The dimension-less functions, $F_1, g_1, g_2$, in the hadron sector are called structure functions which contain all the information of hadron structure.

The inner structure of hadron can be investigate in “hard” scattering kinematic range (DIS range) defined by

$$-q^2 = Q^2 \gg O(p^2) \quad W^2 = (q+p)^2 \gg O(p^2) \quad x_B = \frac{Q^2}{2P \cdot q} \text{ fixed (0 < } x_B < 1),$$  \hspace{1cm} (3.18)

where the dimension-less variable $x_B$ is called Bjorken $x$. The condition of $Q^2$ determines the resolution of the lepton probe for the “inner” structure of the hadron, and $W^2$ condition ensures no resonant state appearing in the photon-target system. In that range of scattering, point-like structure inside of the hadron, i.e. parton, becomes visible. It was firstly observed in SLAC accelerator in late 1960s [30, 31, 32] when the parton picture appeared. The point-like structure was confirmed by the observation that the structure functions of the DIS cross section in eq. 3.17 became finite (non-zero) and (approximately) independent on $Q^2$, called Bjorken scaling. As we have seen in the previous chapter, the scaling breaks slowly as the function of logarithms of $Q^2$. The DIS processes are intensively measured with the lepton accelerators all over the world in wide kinematics range which I will show in the next chapter. The greatest property of perturbative QCD is that we can treat all of these experimental data spreading to extensive range of kinematics on a footage of the parton distributions and the short distance cross sections because it is based on the fundamental property of the field
theory. In the following subsections, I’m going to introduce unpolarized DIS and longitudinally polarized DIS respectively.

### 3.2.1 Unpolarized DIS

Unpolarized DIS is the process of DIS with unpolarized target and unpolarized lepton beam. In this scattering the cross section becomes the function of $F_1$ and $F_2$ as the result of the only contributions from the symmetric ($S$) part of eqs. 3.14 and 3.17. These functions surely survives in DIS region and are investigated to obtain the information on the hadron structure. We can rewrite the cross section in the Lorentz invariant form by defining the variable $y$ as

$$y = \frac{2P \cdot q}{2P \cdot k}, \quad Q^2 = sxy,$$

where $s$ is the square of the center of mass of the lepton-hadron system ($S = (k + p)^2$). Then, the differential cross section on the variable $x_B$, simply $x$, and $y$ can be described by

$$\frac{d^2\sigma}{dxdy} (ep \rightarrow e'X) = \frac{4\pi\alpha^2 s}{Q^4} \left( (1 - y) F_2(x, Q^2) + xy^2 F_1(x, Q^2) \right),$$

where I neglected the term proportional to the mass of the target ($M^2/(s - M^2)$), used the abbreviation $\alpha$ for $\alpha_s$ and rewrite the variables of the dimensionless structure functions in $Q^2$ and $x$. For later use, let me rewrite it to the differential cross section on $x$ and $Q^2$.

$$\frac{d^2\sigma}{dx dQ^2} = \frac{8\pi\alpha^2}{Q^4} \left( \frac{1 - y}{x} F_2(x, Q^2) + \frac{y^2}{2} 2F_1(x, Q^2) \right),$$

$$= \frac{4\pi\alpha^2}{Q^4} \left( (1 + (1 - y)^2)2F_1(x, Q^2) + 2(1 - y)F_L(x, Q^2) \right),$$

where $F_L$ is defined as

$$2F_1 = F_L + \frac{F_2}{x}.$$

In the radiatively corrected parton model description, those structure functions can be calculated following the mass factorization with the diagram like fig. 2.21 or OPE method. (In OPE case, the twist two operators are like those in eqs. 2.10 and 2.11) up to $\alpha_s$ which is needed for NLO calculation,
as

\[ 2F_1(x, Q^2) = \sum_{q, \bar{q}} e_q^2 \left( q(x, \mu_F^2) + \frac{\alpha_s(\mu_F^2)}{2\pi} \left( C^I_q \otimes q(x, \mu_F^2) + C^I_\bar{q} \otimes g(x, \mu_F^2) \right) \right), \]

\[ (3.24) \]

\[ F_L(x, Q^2) = \frac{\alpha_s(\mu_F^2)}{2\pi} \sum_{q, \bar{q}} e_q^2 \left( C^L_q \otimes q(x, \mu_F^2) + C^L_\bar{q} \otimes g(x, \mu_F^2) \right) \]

\[ (3.25) \]

where \( q, \bar{q}, g \) are unpolarized parton distribution functions given in the previous chapter, \( e_q = -e_{\bar{q}} \) is the fractional charge of the quark \( q \) and \( \otimes \) is the convolution integral on \( x \) defined in eq. 2.96. \( \alpha_s \to 0 \) limit ensures the consequence of the naive parton model. Note that \( \alpha_s(\mu^2_R) \) is essentially equal to \( \alpha_s(\mu^2_F) \) through the relation \( \mu_R = c \mu_F \) as discussed in eq. 2.100. (Here and the following, \( \mu^{2}_{0,R} \) and \( \mu^{2}_{0,F} \) which are expected to be the order of \( Q^{2} \) are simply expressed as \( \mu^{2}_{R} \) and \( \mu^{2}_{F} \).) The \( C \)s are the fiction of \( C(x, Q^2/\mu^2_F) \) and called coefficient functions which describes the short distance interaction between virtual photon and partons appearing as the results of the factorization. Note here that the physically observed \( x_B \) can be identified with the partonic \( x \) which expresses the fractional momentum of parton against hadron. The actual form of \( Cs \) can be found, for example, in [75] [138].

The unpolarized cross section or equally structure functions \( F \) is described by unpolarized PDFs. Therefore we can get the information on them from the process. As explained in the previous chapter, the unpolarized PDFs is the distribution functions expressing momentum distribution inside a hadron as the probability densities. As an index of the hadron, it is usual to consider PDFs for the proton. As the momentum probability density, unpolarized PDFs are expected to satisfy the following sum rules.

\[ \int_0^1 dx \left\{ q(x, \mu^2_F) + \bar{q}(x, \mu^2_F) \right\} = q(1, \mu^2_F) + \bar{q}(1, \mu^2_F) = \text{const} \]

\[ (3.26) \]

\[ \int_0^1 dx \left\{ \sum_{q, \bar{q}} q(x, \mu^2_F) + g(x, \mu^2_F) \right\} = \sum_{q, \bar{q}} q(2, \mu^2_F) + g(2, \mu^2_F) = 1 \]

\[ (3.27) \]

where I applied Mellin moments defined in eq. 2.97. Up to NLO, the splitting functions surely satisfy these sum rules, i.e. those values keep unchanged in its scale evolution. These rules are applied as the constraints for the unpolarized PDFs at some initial scale. The proton unpolarized PDFs are extensively investigated applying perturbative QCD and well determines mainly
by virtue of the vast kinematic coverage achieved by HERA accelerator, e.g. \cite{41}. Now unpolarized packages given by CTEQ collaboration \cite{42} and by MRST group \cite{43} are commonly used. Those accurate information on the momentum structure of proton currently became a basic tool especially for studying new physics in the upcoming proton-proton collider, LHC. By the short analysis of the splitting functions, e.g. \cite{73}, in LO which dominates in much higher energy, it turns out that the momentum fraction carried by a quark flavor becomes $1/(16/3+n_f)$ \cite{79} in the asymptotic energy scale. Thus in the limit, the momentum fraction between quark and gluon sectors to be

$$ \text{quarks} : \text{gluon} = \frac{n_f}{16/3 + n_f} : \frac{16/3}{16/3 + n_f}. \quad (3.28) $$

In case $n_f = 6$, roughly the half of the proton momentum is carried by gluons, greatly contrast to quark model. After defining the proton PDFs, those of neutron is expected to be given by interchanging the roles between flavor $u$ and $d$ because of SU(2) flavor symmetry between them in QCD. PDFs of deuteron as a combined state of proton and neutron is naively expected to be the sum of those of proton and neutron. Indeed there are several studies to determine them independently, e.g. \cite{139}, those expectations were directly applied in my analysis and seemed to be consistent enough with existing experimental data.

### 3.2.2 Longitudinally Polarized DIS

Longitudinally polarized DIS, simply polarized DIS in the following, is the DIS process with longitudinally polarized lepton and longitudinally polarized target. The cross section of the process contains the contributions coming from the antisymmetric term ($A$) in eqs. 3.14 and 3.17. Experimentally, it is hard work to entangle the kinematics which appears in the cross section of this process because of the contributions from the symmetric part. Thus to isolate the anti-symmetric contribution, the difference of the cross sections are usually taken;

$$ d\sigma_{LL} = \frac{1}{2}(d\sigma^{\overleftrightarrow{\gamma}} - d\sigma^{\overleftrightarrow{\gamma}}), \quad (3.29) $$

where the double arrow indicates target helicity, and the other is that of lepton. The symmetric part drops and the anti-symmetric part only remains. Then the differential cross section on $x$ and $Q^2$ can be described as

$$ \frac{d^2\sigma_{LL}}{dx dQ^2} = \frac{8\pi\alpha^2 y}{Q^4} \left( \left( 1 - \frac{y}{2} - \frac{y^2}{4} \gamma^2 \right) g_1(x, Q^2) - \frac{y}{2} \gamma^2 g_2(x, Q^2) \right), \quad (3.30) $$
where $\gamma^2 = Q^2 M^2 / (p \cdot q)^2$. In the language of OPE, the structure function $g_1$ is described with operators starting from twist two, whereas $g_2$ expression is known to be given by those of twist three operators at least [140]. Thus in the kinematics of our interest, $g_2$ becomes inferior in the contribution to the cross section and $g_1$ dominates. In fact, the study of $g_2$, related to the transversely polarized PDFs or other higher twist distributions, has also an importance for the comprehensive understanding of hadron spin structure [119, 125]. In my study, however, I’m going to treat them as a marginal contribution to the longitudinally polarized cross section. The twist two local operators $O_1$ for $g_1$ are given [140, 141] as

$$O_1^{F} = \bar{q} S \langle \gamma_5 \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} \rangle q - \text{traces},$$

$$O_1^{V} = \bar{q} S \langle F_{a_1}^{\mu_1 \nu_1} D_{a_2} \cdots D_{a_{n-1}} F_{a_n}^{\mu_n - 1 \nu} \rangle - \text{traces}$$

(3.31) (3.32)

where the notations are the same as eqs. 2.10 and 2.11 and eq. 2.131 is applied. If we notice that the correspondence between the definition of unpolarized PDFs, eqs. 2.123 and 2.128 and the equivalent local operators, eqs. 2.10 and 2.11 we can guess that the $g_1$ is related to the parton helicity distributions, defined in eqs. 2.129 and 2.130. Actually, the parton level calculation of $\sigma_{++} - \sigma_{+-}$ in the mass factorization method given in the previous chapter with the projection operators $\Delta U, \Delta L$ yields the direct calculation of (only) $g_1$ part and the factorization of $g_1$ with the helicity distributions, fig. 3.3. (Note that the mass factorization method is only focusing on the extraction of the lowest twist contribution.) Note however that the first + sign of the $\sigma$ indicates the helicity state of virtual photon. Thus it is expected that we have simpler form if we obtain the difference cross section against virtual photon polarization. Experimentally, to make its analysis simpler, the quantity actually measured is the asymmetry given by the fraction of the difference and the sum of the differential cross sections of the two polarization states;

$$A_1 = \frac{d\sigma^{+} - d\sigma^{-}}{d\sigma^{+} + d\sigma^{-}},$$

(3.33)

Then the asymmetry for the virtual photon polarization, named $A_1$, surely has simple form [142, 125] of

$$A_1 = \frac{g_1(x, Q^2) - \gamma^2 g_2(x, Q^2)}{F_1(x, Q^2)},$$

(3.34)

where $F_1$ is one of the unpolarized structure function defined above. The actual extraction of $A_1$ from a measured asymmetry with lepton polarization (not with virtual photon), referred to as $A_{\parallel}$, can be found in the above
Figure 3.3: pictorial expression of the calculation of $g_1$ part: ± denotes helicity state of each particle, refer also to fig. 2.24.

references or the papers from experiments [143] [50]. Thus if we neglect the marginal contribution from $g_2$, we can simply compare the parton description of $g_1$ with measured $A_1$.

As the result of the partonic expression of $g_1$ we can obtain the following factorized expression.

$$2g_1(x, Q^2) =$$
$$\sum_{q,\bar{q}} e_q^2 \left( \Delta q(x, \mu_F^2) + \frac{\alpha_s(\mu_F^2)}{2\pi} \left( \Delta C_q^1 \otimes \Delta g(x, \mu_F^2) + \Delta C_q^2 \otimes \Delta g(x, \mu_F^2) \right) \right),$$

(3.35)

where $\Delta C(x, Q^2/\mu_F^2)$s are the (polarized) short distance cross section calculable also in the mass factorization method with the helicity projection operators $\Delta U, \Delta L$. The actual form of them can be found in [111] [118]. From the naive quark model, eq. 1.6 and the naive parton model, $g_1$ or the cross section eq. 3.29 is expected to be positive in the proton target case because of the up quark dominance, and negative in the neutron case because the rich down quark compensates the difference of the squared charge.
Thus, from longitudinally polarized DIS, we can access information on the hadron structure thorough the helicity distributions like unpolarized DIS. As mentioned in the previous chapter, the helicity distributions has also the interpretation of the probability density, but the difference of the helicity state densities, \( \text{fig. 2.25} \) (Then the positivity condition, eq. 2.139 is imposed.) So the helicity distributions express the spin bias or the spin contribution to the spin of the hadron. Like unpolarized PDFs, the proton is taken as the standard hadron. As the helicity probability density, the helicity distributions are expected to answer the following sum rules

\[
\Delta q(1, \mu_F^2) + \Delta \bar{q}(1, \mu_F^2) - (\Delta q'(1, \mu_F^2) + \Delta \bar{q}'(1, \mu_F^2)) = \text{const} \tag{3.36}
\]

: fermion helicity conservation,

\[
\int_0^1 dx \left( \frac{1}{2} \sum_{q, \bar{q}} \Delta q(x, \mu_F^2) + 1 \cdot \Delta g(x, \mu_F^2) \right) + L_{q+g} = \frac{1}{2} \sum_{q, \bar{q}} \Delta q(1, \mu_F^2) + \Delta g(1, \mu_F^2) + L_{q+g} = \frac{1}{2} \tag{3.37}
\]

: angular momentum (spin) sum rule,

where \( L_{q+g} \) represents the contributions of parton orbital angular momentum. Like unpolarized case, the polarized splitting function, e.g., \([118]\) guarantees the helicity conservation at least up to NLO. This comes from the fact that the non-singlet splitting function \( \Delta P_{NS}^+ \) defined in eq. 3.6 satisfies \( \Delta P_{NS}^+ = P_{NS} \Delta P_{VV}^+ = -P_{VV}^+ \) as a consequence that the projection operators for the anti-quarks \( \Delta U_{\bar{q}} \) and \( \Delta L_{\bar{q}} \) changes its sign (eq. 3.26 or eq. 3.23). Whereas, unlike unpolarized case, the angular momentum sum rule contains the term which cannot be described by the helicity distributions. The similarity with the unpolarized case urges us to define the distribution related to the total angular momentum. Because the idea of the angular momentum is three dimensional one, we have to extend our distribution to have the information also on the transverse direction to parton momentum. Actually it is achieved, and more general distribution functions, called \textit{generalized parton distributions} (GPDs) or \textit{off-forward parton distributions} (OFPD), are defined \([144, 145, 54]\). The total spin structure (including momentum structure) of a hadron can be comprehensively understood with those distributions. I don’t go into its detail, but using those distributions, we can define the total angular momentum of a parton, \( J_q(\mu_F^2) \). Then the proton spin sum rule can be simply rewritten as

\[
\sum_{q, \bar{q}} J_q(\mu_F^2) + J_g(\mu_F^2) = \frac{1}{2} \tag{3.38}
\]
Just as the unpolarized case, those fractions in the asymptotic energy scale again becomes

\[ \text{quarks : gluon} = \frac{n_f}{16/3 + n_f} : \frac{16/3}{16/3 + n_f}, \]  

which was firstly pointed out in [144]. The investigation of hadron structure based on the general distributions has just started experimentally using a process sensitive to those distributions, e.g. deeply virtual Compton scattering [146, 147]. Thus for the now coming investigation, there would be important to pin down the parton helicity structure firmly and to make those results a mile stone for better understanding of the rich inner structure of hadrons. This is the main issue of this thesis. An Example of the packages currently often referred to is that by GRSV group [62]. Just recently the results of the analysis with the same idea with us to handle more comprehensive experimental data appeared [56].

### 3.3 Inclusive Hadron Production in $e^+ e^-$ Collision

Inclusive hadron production in $e^+ e^-$ collision, also called single hadron inclusive measurement in $e^+ e^-$ annihilation (SIA), is the process of electron-positron annihilation to jet of a measured hadron and others, fig. 3.4

\[ e(k\lambda) + \bar{e}(k'\lambda') \rightarrow H(p, \sigma) + X. \]  

In the following I treat the process with unpolarized leptons in the initial state and unpolarized measured hadron in the final state, and consider the process in the center of mass system of the leptons. For the polarized case, we can refer to [148, 120] for example. The differential cross section $d\sigma^H$ of this process with lepton momentum $k' = (Q/2, 0, 0, (-)Q/2)$ is described as a function of $Q^2$ and $z$ which are defined as follows.

\[ q = k + k', \quad Q^2 = q^2 = s, \]
\[ z = \frac{2p \cdot q}{q^2} = \frac{2p_0}{Q}. \]

Then the cross section is described as

\[ \frac{d\sigma^H}{dz} = \frac{4\pi\alpha_e}{s} \left( 2F_1^H(z, Q^2) + F_L^H(z, Q^2) \right), \]
where $F_{1,L}$ are dimensionless structure function which contains all the information on the hadronization \cite{149,66}. Experimentally, the fraction

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma^H}{dz}$$

(3.44)

is usually provided where $\sigma_{\text{tot}}$ is defined as the total cross section of hadronic process in $e^+ e^-$ collision expressed up to $\alpha_s$ as

$$\sigma_{\text{tot}} = \sum_q \hat{e}_q^2 4\pi\alpha_e \frac{s}{\pi} \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right)$$

(3.45)

where $\hat{e}_q$ is the effective fractional charge defined below.

The perturbative QCD can be applied to the process in which the center of mass energy $\sqrt{s}$ of the process much higher than detected hadron mass. Then the time-like diagram, fig. 3.5 serves as the parton description of the process along with the factorization. Thus through this process we can purely access the (unpolarized) fragmentation functions defined in the previous chapter. Then the $F^H_1(z,Q^2)$ and $F^H_L(z,Q^2)$ is described with the
fragmentation functions $D^H$ up to $\alpha_s$ as

$$2F_1^H(z, Q^2) = \sum_{q,\bar{q}} \hat{e}_q^2 \left( D_q^H(z, \mu_F^2) + \frac{\alpha_s(\mu_F^2)}{2\pi} \left( C_{q,1}^{H,1} \otimes D_q^H(z, \mu_F^2) + C_{g,1}^{H,1} \otimes D_g^H(z, \mu_F^2) \right) \right),$$

$$(3.46)$$

$$F_L^H(z, Q^2) = \frac{\alpha_s(\mu_F^2)}{2\pi} \sum_{q,\bar{q}} \hat{e}_q^2 \left( C_{q,1}^{H,L} \otimes D_q^H(z, \mu_F^2) + C_{g,1}^{H,L} \otimes D_g^H(z, \mu_F^2) \right),$$

$$(3.47)$$

where again $z$ in the structure functions is replaced on the right hand side by the partonic $z$ which expresses momentum inverse fraction against the final state parton momentum. Again the $C$s are functions of $C(z, Q^2/m_F^2)$, and represent the partonic short distance cross section. The actual form can be found, for example, in [138, 148]. The final remark is the effective fractional charge $\hat{e}_q$. It appears as the result of the mixing of $Z$ boson around the $Z$ mass scale and is defined as

$$\hat{e}_q^2 = e^2 - 2e_q\chi_1(Q^2)V_eV_q + \chi_2(Q^2)(1 + V_e^2)(1 + V_q^2),$$

$$(3.48)$$

$$\chi_1(s) = \frac{1}{16 \sin^2\theta_W \cos^2\theta_W} \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + \Gamma_Z^2 M_Z^2},$$

$$(3.49)$$

$$\chi_2(s) = \frac{1}{256 \sin^4\theta_W \cos^4\theta_W} \frac{s^2}{(s - M_Z^2)^2 + \Gamma_Z^2 M_Z^2},$$

$$(3.50)$$

Figure 3.5: diagrammatic expression of SIA process
where $\theta_W$ is the Weinberg angle, $M_z$ and $\Gamma_Z$ are the mass and the decay width of the $Z$ boson, appearing from the fermion loop corrections to the $Z$ boson propagator \([150]\), and $V_{e,q}$ are given

$$V_e = -1 + 4 \sin^2 \theta_W \ , \quad (3.51)$$

$$V_u = +1 - \frac{8}{3} \sin^2 \theta_W \quad \text{up-type quark} \ , \quad (3.52)$$

$$V_c = -1 + \frac{4}{3} \sin^2 \theta_W \quad \text{down-type quark} \ . \quad (3.53)$$

The second term in eq. \([3.48]\) corresponds to the interference term between the virtual $\gamma^* \text{ and } Z^*$, and it vanishes at $s = M_z^2$. At the $s \sim M_z^2$ scale, the first term gives only few present contribution because of the $Z$ resonance. Thus the treatment of the proper propagator, which results in “charge” this time, is mandatory for quantitative expression of the cross section around the kinematic range. (Actually I will include such data in my analysis.)

The fragmentation functions satisfies the following sum rules \([149]\).

$$D_q^H (1, Q^2) - D_{\bar{q}}^H (1, Q^2) = \text{const} \quad (3.54)$$

: flavor conservation.

$$\sum_H D_q^H (2, Q^2) = 1 \quad (3.55)$$

: energy momentum conservation.

Up to NLO, the splitting functions surely conserve these sum rules. Several analysis groups did their analysis in the perturbative QCD framework to extract those functions. The standard ones are those by Kretzer \([66]\) and KKP group \([151]\). The new fragmentation function package \([69, 68]\) based on the same analysis ideology with us appeared recently. In my analysis, fragmentation functions to $\pi^\pm$, $K^\pm$ and $h^\pm$ are treated. Here $h^{+(-)}$ is the sum of positively (negatively) charged hadrons. The assumption which are expected to be satisfied by those functions are the charge symmetry.

$$D_q^H = D_{\bar{q}}^H \ , \quad D_{\bar{q}}^H = D_q^H \ , \quad (3.56)$$

If necessary, those of neutral pion $\pi^0$ are provided by

$$D_{q}^{\pi^0} = \frac{1}{2} \left( D_{q}^{\pi^+} + D_{q}^{\pi^-} \right) \ . \quad (3.57)$$
3.4 Semi-Inclusive Measurement of Deep Inelastic Scattering

In the following, I’m going to introduce the kinematics of the processes which includes several distributions. Semi-inclusive DIS (SIDIS) is the DIS scattering accompanied by additional hadron detection in the final states, fig. 3.6.

\[ e(k, \lambda) + H'(p, \sigma) \rightarrow e'(k', \lambda') + H(p', \sigma') + X. \]  

(3.58)

What we are currently interested in are the processes without detection of the polarization of final state hadron. The tensorial structure of this process is exactly the same as DIS case, eqs. 3.12 - 3.17 but with replacing the structure functions \( F, g \) to \( F^H, g^H \) which depend not only \( x \) and \( Q^2 \), but also the new variable \( z \) respecting the additional hadron detection [85, 138, 148];

\[ z = \frac{p \cdot p'}{p \cdot q}, \]  

(3.59)

which has the meaning of the energy fraction of the observed hadron over the energy of the virtual photon. Except the replacement and the additional differentiate on \( z \), the cross section expressions, eqs. 3.22, 3.30 and 3.34 hold for SIDIS process, e.g. for unpolarized SIDIS, eq. 3.22 becomes

\[ \frac{d^2\sigma^H}{dx Q^2 dz} = \frac{8\pi\alpha^2}{Q^4} \left( \frac{1-y}{x} F^H_2(x, z, Q^2) + \frac{y^2}{2} 2F^H_1(x, z, Q^2) \right). \]  

(3.60)

![Figure 3.6: kinematics of SIDIS process](image)

The dimensionless structure functions \( F^H, g^H \) can be calculable with the
It gives the parton model interpretation to those functions with the factorization. The functions are described in a form like

\[ 2g_1^H(x, z, Q^2) = \sum_{q, \bar{q}} e_q^2 \left( q(x, \mu_F^2) D_q^H(z, \mu_F^2) + \frac{\alpha_s(\mu_F^2)}{2\pi} \left( \Delta C_{qq}^1 \otimes_x \Delta q(x, \mu_F^2) \otimes_z D_q^H(z, \mu_F^2) + \Delta C_{gq}^1 \otimes_x \Delta g(x, \mu_F^2) \otimes_z D_g^H(z, \mu_F^2) \right) \right) , \]

where \( \mu_F^2 \) and \( \mu_F'^2 \) are factorization scales of PDFs and FFs respectively, the coefficient functions, \( C_{1,2,3} \), are functions of the form \( C(x, z, Q^2/\mu_F^2, Q^2/\mu_F'^2) \) which are independent of the hadron type final state \( H \) as its short distance nature, and \( \otimes_{x,z} \) represents the convolution integral, eq. 2.96 on the variable \( x \) or \( z \). Note here that I explicitly indicate \( \mu_R^2 \) dependence of \( \alpha_s \) to show that it is essentially consistent object through the relation \( \mu_R = c\mu_F = c'\mu_F' \).

The other structure functions, \( F_{1,2,3}^H \), have the same expansion with helicity distributions, \( \Delta q(x) \)s, replaced by unpolarized PDFs, \( q(x) \)s and the polarized coefficient functions, \( \Delta C \)s, by those of unpolarized one. Concrete expression of the coefficient functions, \( C^H \)s, can be found in the appendix of [148].
The important observation is that the parton distributions and fragmentation functions which appeared in eq. 3.64 are identical with those emerged in DIS and SIA. As it is clear from those derivation in the previous chapter, those distributions are defined from the behavior of partons in IR divergent (long distance) region, and those IR nature of them is quite general, so independent on the process in which the partons are included as we saw in the previous chapter. This is called “universality” of distribution functions, and one of the key observations in perturbative QCD framework.

SIDIS quantities I’m going to include are SIDIS spin asymmetry $A_H^1$ and multiplicity $\eta^H$. The asymmetry $A_H^1$ is defined similarly as eq. 3.34 calculable with $g_1^H$ and $F_1^H$ with $g_2^H$ neglected again. The multiplicity $\eta^H$ is defined as the fraction of the differential cross section of unpolarized DIS, eq. 3.22 and that of unpolarized SIDIS, eq. 3.60 both of which come along with the parton description.

$$\eta^H = \frac{d^2\sigma^H}{dx dQ^2 dz}(x, z, Q^2)/\frac{d^2\sigma}{dx dQ^2}(x, Q^2).$$  (3.65)

The biggest advantage of including these processes is that we can effectively separate the contributions from quark distributions and anti-quark distributions. As we see in eqs. 3.35, 3.25 and 3.47, although those pure process provides clear view on each type of distributions, we cannot approach in principle to the information on the difference of them because of the charge equivalence $|e_q| = |\bar{e}_q|$. (Refer also to the crossing in appendix A) When we have $W$ bosons mediate process, we can by their different coupling to $W^\pm$. Actually in unpolarized DIS in HERA kinematic range, the process occurs and gives clear description on unpolarized PDFs [41]. However in polarized case, currently we don’t reach such kinematic range. While in the process of SIDIS eq. 3.64 the separation is expected to become possible through the existence of the other coupled distributions which are in general different between quark and anti-quark. Thus with the difference of the weight of the contributions in SIDIS process and in the pure one, we expect to obtain the detailed information on the distributions. The firm determination of helicity distributions would also become important for the forthcoming programme of $W$ boson process [52, 152] and verifying the consistency of parton description.

### 3.5 Proton-Proton Collisions

Proton-proton collision provides another processes with multi distributions which is extensively studied in RHIC accelerator for helicity distributions
In the following I’m going to introduce briefly some processes I will use as a prospect from the results of our analysis.

### 3.5.1 Drell-Yan process

Drell-Yan process [153, 154] is the proton-proton scattering with production of hard lepton pair in the final state, fig. 3.8.

\[ p(p, \sigma) + p'(p', \sigma) \rightarrow \gamma^*(q) + X \rightarrow l(k) + \bar{l}(k') + X. \]  

(3.66)

I’m going to concentrate on the virtual photon exchange. The cross section

\[ d\sigma \left( \frac{d\sigma}{dM^2} \right) dy = \frac{4\pi\alpha^2}{9M^2\sqrt{S}} \sum_q \left( \frac{q(x_1^0)\bar{q}(x_2^0) + q(x_2^0)\bar{q}(x_1^0)}{q_0 + q_z - q_0 - q_z} \right)^2 \]  

(3.70)

where \( q \) denotes the four-momentum of the lepton pair in the proton-proton center of mass system. In the parton model, the differential cross section of unpolarized Drell-Yan process on these values has the form.

Figure 3.8: kinematics of Drell-Yan process

\[ y \equiv \ln \left( \frac{1 + \beta}{1 - \beta} \right) = \frac{1}{2} \ln \left( \frac{q_0 + q_z}{q_0 - q_z} \right) \quad (\beta = q_z/q_0), \]  

(3.67)

\[ M^2 \equiv (k + k')^2 \quad : \text{invariant mass of the lepton pair}, \]  

(3.69)

: rapidity of the virtual photon,

where \( q \) denotes the four-momentum of the lepton pair in the proton-proton center of mass system. In the parton model, the differential cross section of unpolarized Drell-Yan process on these values has the form.

\[ \frac{d\sigma}{dM^2} \]  

(3.70)
where $x_0^1$ and $x_0^2$ are given with $\tau \equiv M^2/S$ as

$$x_0^1 = \sqrt{\tau} e^y, \quad x_0^2 = \sqrt{\tau} e^{-y}, \quad (3.71)$$

which have the interpretation as the momentum fractions of the quarks in the original proton noting $x_0^1, x_0^2 = \tau$ and $p^0 = (\sqrt{S}/2, 0, (-)\sqrt{S}/2)$. The scaling can be observed as small logarithmic dependence of the cross section on $\tau$ as obvious from eq. (3.70). Because of the direct coupling of quark and anti-quark distribution, we can access poorly known sea quarks (anti-quarks) through this process.

The partonic cross section eq. (3.70) can be modified by the QCD radiative correction \[.155\]. The diagrammatic expression of this modification is described as fig. 3.9. The modification can be performed based on the framework introduced in the previous chapter. Up to NLO of perturbative QCD, the differential cross section for unpolarized Drell-Yan is modified as

$$\frac{d\sigma}{dM^2dy} = \frac{4\pi \alpha_s^2}{9M^2S} \sum_q e_q^2 \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2$$

$$\left\{ \left[ \frac{d\hat{\sigma}_{q,\bar{q}}^{(0)}}{dM^2dy} + \frac{\alpha_s(\mu^2_R)}{2\pi} \frac{d\hat{\sigma}_{q,\bar{q}}^{(1)}}{dM^2dy} \right] \{ q(x_1, \mu_F^2) \bar{q}(x_2, \mu_F^2) + (1 \leftrightarrow 2) \} + \right. \left. \left\{ \frac{\alpha_s(\mu^2_R)}{2\pi} \frac{d\hat{\sigma}_{g,g}^{(1)}}{dM^2dy} q(x_1, \mu_F^2) \{ q(x_2, \mu_F^2) + \bar{q}(x_2, \mu_F^2) \} + (1 \leftrightarrow 2) \right\} \right\}, \quad (3.72)$$

Figure 3.9: pictorial diagram of pQCD correction for Drell-Yan process.
where $\hat{\sigma}$ is the function of $\hat{\sigma}(x_1, x_2, M^2/\mu_F^2)$ which means the partonic cross section between partons with momentum fractions $x_1$ and $x_2$ to the original protons, and I took the identical factorization scale $\mu_F^2$ for the distributions coming from the different protons for simplicity. The extension to the expression in longitudinally polarized case is obvious. The integration of the above expression could not reduces to the convolution integral appeared everywhere in the processes introduced as far. This is because the cross section contains the angle variable $y$ unintegrated. The convolution integral appearing in the process of the factorization comes from the fact that we always took the spatial phase integral. Thus if we integrate over the rapidity $y$, we can surely have the factorized form described by the convolution integral\cite{138,150}. However consider the actual application of the description, it would be realistic to keep $y$ dependence explicitly. For concrete form of the partonic cross sections for $d\hat{\sigma}$ up to NLO, refer for example to \cite{157} for unpolarized case and \cite{156} for longitudinally polarized case. For other attractive processes which contain multi distributions, we can refer to \cite{158,159,160,161}.
Chapter 4

Data Analyses

In this chapter, I’m going to indicate the methods of my analysis based on perturbative QCD. In the first section, the key ingredient of this analysis, Mellin transformation, is discussed. Then I develop the detailed formalism for the calculations based on the Mellin transform. The implemented ingredients in our framework will also be listed there. In section 4.2, I will take some space for the discussion on $\chi^2$ fit because several arguments occurred between helicity distribution analysis groups before. Thus it would be good to clear up the meaning of $\chi^2$ fit. The definitions of statistical errors and correlations given in my analysis will be provided there. In section 4.3, I will concede an overview of our calculation and fit framework. In the proceeding sections, I’m going to describe several conditions I imposed in my analysis. Finally in section 4.7, I listed up the systematic errors which are considered to exist in our analysis.

4.1 Mellin Transformation

Mellin transformation is the crucial improvement in our successive analysis strategy. In our earlier study for helicity distribution and fragmentation functions \[58, 57\], the more intuitive method, named brute force \[162, 163\], was used. In that framework, we encountered lengthy computation time mainly coming from the convolution integral appearing everywhere in perturbative QCD results. It became much serious when the analyses went to handling more complicated processes, like SIDIS process, and tortured us in performing detailed analyses without losing the accuracy of the results. To avoid a barrier of the computation time, we construct a new analysis structure based on a method, called Mellin transformation. By virtue of the transformation, the detailed study recently became ready to be performed in our framework.
within a compatible time scale. In this section, I’m going to introduce the idea of the method.

4.1.1 Basic Concepts

The Mellin transformation is the integral transformation defined by

\[ f(n) = \int_{0}^{1} dx \, x^{n-1} f(x), \tag{4.1} \]

which we have already encountered several times in the context of mass factorization and operator product expansion in chapter 2. As already mentioned in eq. 2.97, the most profitable feature is that the convolution integral

\[ \int_{x}^{1} dy \frac{dy}{y} f\left(\frac{x}{y}\right)g(y) = \int_{0}^{1} \int_{0}^{1} dz dy f(z)g(y) \delta(x - zy) = f \otimes g \tag{4.2} \]

reduces to simple multiplication when the moments are taken in both sides. From the point of view of actual analysis, the simplification of the time consuming integral is very attractive.

The property of the Mellin transformation becomes obvious when it ascend back to Laplace transform as a special case of Fourier transform. The Laplace transform is defined as

\[ F(s) = \int_{0}^{\infty} dt e^{-st} f(t) \tag{4.3} \]

where \( f(t) \) is a function defined in \((0, \infty)\) and \( s \) is a complex value \( s = \sigma + i\tau \). The convergence of the integral depends on the value \( s \). If the integral converges at a complex value \( s_0 \), then \( F(s) \) converges at arbitrary value of \( s \) if \( \text{Re}(s) > \text{Re}(s_0) \). Thus we can find a real value \( \gamma_0 \) so that \( F(s) \) converges at arbitrary \( s \) for \( \text{Re}(s) > \gamma_0 \). The minimum of \( \gamma_0 \) is called convergence coordinate, e.g. in case \( f(t) = 1 \), \( \gamma_0 = 0 \) and then \( F(s) \) becomes \( 1/s \). Additionally, for normally used functions, the converged \( F(s) \) behaves as \( \lim_{s \to \infty} F(s) = 0 \). When \( F(s) \) is analytically continued to the full complex plane, this can be seen as \( F(s) \) is regular in the region \( \text{Re}(s) > \gamma_0 \), whereas \( F(s) \) may have poles or brunch cuts in its left, fig. 4.1. In such a convergence region, just like Fourier transform, there is an inverse of the Laplace transform, inverse Laplace transform, defined as

\[ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} ds e^{st} F(s) = \frac{1}{2} \{ f(t + 0) + f(t - 0) \} \tag{4.4} \]
where $\gamma > \gamma_0$ and $f(t \pm 0)$ indicates approaching $t$ from $\pm$ direction which is needed for the case $f(t)$ is discrete at $t$. This integral returns 0 for $t < 0$ because of the regularity of $F(s)$ in the right region. Note that, from the viewpoint of the analytic behavior of $F(s)$, the complex integration can be deformed in another path as far as $F(s)$ is regular in the deformed area because of Cauchy theorem and the fact $\lim_{s \to \infty} F(s) = 0$, fig. 4.1.

When we perform a replacement $t = -\ln x$, we can easily reproduce the Mellin transform eq. (4.1). (Note that $(0, \infty)$ in $t$ becomes $(1, 0)$ in $x$.) Then we notice that we can also define inverse Mellin transform as

$$f(x) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} dn x^{-n} f(n).$$  \hspace{1cm} (4.5)

where $c_0$ is a real value to be taken as all of the singularities of $f(n)$ locates its left, and I ignored discontinuity of $f(x)$. As it would be clear from the definition, $n \to \infty$ corresponds to $x \to 1$ in $x$ space, and $n \to 0$ corresponds to $x \to 0$. Note that, as the Laplace transform case, we can also deform the path in regular region of $f(n)$. This arbitrariness in choosing the path of integral is made use of in our actual numerical calculations to obtain a good convergence.
\[
\begin{array}{|c|c|}
\hline
f(x) & f(n) \\
\hline
x^\alpha & \frac{1}{n+\alpha} \\
\delta(1-x) & 1 \\
\ln x & \frac{1}{n^2} \\
\ln(1-x) & -\frac{1}{n} S_1(n) \\
\frac{\ln x}{1-x} & -\frac{1}{6} \pi^2 - \frac{1}{n} + S_1(n) \\
\frac{1}{(1-x)_+} \left( \frac{\ln(1-x)}{1-x} \right) & \frac{1}{n} - S_1(n) + \frac{1}{2} S_1(n)^2 + \frac{1}{2} S_2(n) - \frac{1}{n} S_1(n) \\
\hline
\end{array}
\]

Table 4.1: example of Mellin moments

### 4.1.2 Application to perturbative QCD Calculation

The prominent feature of the simplification of the convolution integral and existence of its inverse urges us to the application of Mellin transform method to perturbative QCD (pQCD) analysis. Moreover, the analytic behavior of theoretical results is guaranteed from the diagrammatic treatment of the mass factorization as I mentioned in the previous section. Thus the method of Mellin transform seems to be nicely suited to analyses based on pQCD. (Indeed there are other methods like \[164, 165\].) Now the most of the pQCD analysis groups employ this method especially for polarized PDF \[166, 167\] for the complexity of the physical processes. In this subsection, I’m going to introduce some basic formulas needed to actual application of the Mellin transform to pQCD analysis.

Fortunately the splitting functions needed for DGLAP evolution are known to be able to be converted to the complex space, moment space, analytically using some special functions \[168\] at least NLO. The table 4.1 is a list of some examples of the transformation. In the table, \( S_k(n) = \sum_{i=1}^{n} 1/k^j \) and \( S_j(n) \) are analytically connected to complex plane using poly gamma functions \[169, 170\]. Then the DGLAP equations, eqs. 3.3 and 3.8 reduces to matrix differential equations. The actual implementation of the equation is given in the next subsection. For some short distance coefficient functions, their analytic forms are also known \[168, 66, 121\] up to NLO level. Thus pQCD predictions or calculations of several observable quantities can be consistently handled in the calculation framework based on Mellin transform.
To make prediction on x space, we have to take an invert of the moment space quantities using inverse Mellin transformation. The choice of the path is arbitrary in principle if the poles of splitting functions or coefficient functions are on its left plane. Because of its physical meaning as splitting or short distance cross section, those functions are real functions of x. Thus the poles exist only on the real axis, and satisfy $f^*(n) = f(n^*)$, fig 4.2. Thus we can treat only the integral in the above half plane. In case of the original

Figure 4.2: contour of inverse Mellin transform

contour, which is parallel to the imaginary axis applied in [171], $x^{-n}$ factor is purely oscillating; it does not damp the integrand $n \to \infty$ (indeed $f(n)$ itself is expected to be damped). Considering actual calculations with finite accuracy, we want to have as fast convergence as possible to make faithful calculations. The key issue is the deform-ability of the contour. One technique is deforming the path to lean to negative real axis to make $n$ have an increasingly negative real part as it heads off to infinity. This provides exponential suppression and improves the convergence of the integral [172], fig 4.2. Another fascinating technique is finding the steepest descent path of the integral. The steepest descent curve is a path passing a saddle point of the integrand and keeping the phase of integrand stationary, fig 4.3. Because of the nature of saddle point, then the integrand is expected to fall off rapidly. The idea of the steepest descent itself is applied usually to deduce Starling formula for gamma function from its integration form. Actually the path determination based on the idea of the steepest descent was provided in
finding the saddle point on the real axis and estimate the path keeping the imaginary part of the integrand zero. We implemented both of two contour methods in our calculation framework. However, the latter encountered difficulty in the NLO calculations. Additionally it is not straightforward to apply the method to polarized process according to the difficulty of finding the saddle point on the real axis. (It originates from the fact that the structure functions or PDFs in polarized case can be negative.) Thus our analysis for this thesis is in principle based on the first method of sloping straight line. The steepest descent method was used only for the adequacy checking of the calculation results. The farther investigation on the steepest curve which seems more sophisticated is left for the future study.

To be more specific, let me show the actual quantity to be calculated based on the leaning path line. In case of one inverse integration, like PDFs or DIS, the integral can be written as follows using the mirror symmetry on the above and down half;

$$ f(x) = \frac{1}{2\pi i} \int_{C_{s} + (-\bar{C}_{s})} d_n x^{-n} f(n) = \frac{1}{2\pi i} \int_{C_{s}} (d_n x^{-n} f(n) - d\bar{n} x^{-\bar{a}} f(\bar{n})) $$

$$ = \frac{1}{\pi} \int_{C_{s}} \text{Im} (d_n x^{-n} f(n)) $$

where $C_{s}$ is the upper part of the contour, $\bar{a}$ is the complex conjugate of $a$, and $f(\bar{n}) = f(n)$. In the straight line case, we can parametrize $n$ with
abscissa $c_0$ and the angle $\phi$ as in fig. 4.2

$$n(r) = c_0 + re^{i\phi}. \quad (4.8)$$

Then the integral eq. 4.7 can be rewritten as an integration over $r$.

$$f(x) = \frac{1}{\pi} \int_0^\infty dr \Im \left( e^{i\phi x} e^{-n(r)} f(n(r)) \right). \quad (4.9)$$

In case of double inverse Mellin, fig. 4.4 needed for calculation of SIDIS process, the discussion goes as well, and then we obtain

$$f(x, z) = \left( \frac{1}{2\pi i} \right)^2 \int_{C'_z} dm \int_{C'_s} dn x^{-n} z^{-m} f(n, m) \quad (4.10)$$

$$= -\frac{1}{2\pi^2} \int_{C'_z} \int_{C'_s} \Re \left\{ dn dm x^{-n} z^{-m} f(n, m) - dn d\bar{m} x^{-n} z^{-\bar{m}} f(n, \bar{m}) \right\}, \quad (4.11)$$

where $C'_s$ is the contour for $m$ integration independent from $C_s$, and $\int f(n, \bar{m}) = f(\bar{n}, \bar{m})$ and $\int f(n, m) = f(\bar{n}, m)$ are put in use. Then again we take the straight lines parametrized as

$$n(r) = c_0 + re^{i\phi}, \quad m(r') = c'_0 + r'e^{i\phi'}. \quad (4.12)$$

This parametrization yields the integrations on real variables;

$$f(x) = -\frac{1}{2\pi^2} \int_0^\infty dr' \int_0^\infty dr \Re \left\{ e^{i\phi x} e^{-n(r')} \left( e^{i\phi x} e^{-m(r')} f(n(r), m(r')) - e^{-i\phi'} x^{-\bar{m}(r')} f(n(r), \bar{m}(r')) \right) \right\}. \quad (4.13)$$

(4.14)

After all, for integrations eqs. 4.9 and 4.14 the parameters we have to adjust are length of $r$, angle $\phi$ and offset $c_0$. Those values are currently artificially adjusted to achieve reasonable computation time without loosing the accuracy of the calculation. Moreover, to avoid loosing accuracy as far as possible, we employed integration method Gauss-Legendre integration [175]. In our framework, first we set the accuracy index value and two sets of Gauss-Legendre integrations which defers its orders, then recursively divide an integration area until the integrations by two different order integration in that divided area become consistent within the accuracy of the index value. By this method, we can fulfill highly accurate integration with the ambiguity of order of adjustable precision. Thus for the integration we have tens of
parameters to be adjusted. To find the true optimized values is one of our future issues. Currently however we employed empirical values comparing results between different set of parameters and the results from our $x$ space integral. The consideration of the ambiguities coming from choosing those parameters is included as systematic errors of our results. To eliminate those unsettled ambiguities, the progress on the study of steepest descent would be significant.

Finally, I would like to comment on another advantage to employ Mellin moment calculation. In actual experimental data, some data are given as accumulated yields over certain bins. For comparison with those experimental data, we have to take integral on those bins. In Mellin formalism, the moment can be easily calculated without losing computation time as follows;

$$
\langle A(x) \rangle_m = \int_{x_L}^{x_H} dx x^m A(x) = \frac{1}{2\pi i} \int_C d\eta \int_{x_L}^{x_H} dx x^{-n+m} A(\eta) \quad (4.15)
$$

$$
= \frac{1}{2\pi i} \int_C d\eta \frac{x_H^{1-n-m} - x_L^{1-n+m}}{1 - n + m} A(\eta). \quad (4.16)
$$

The extension to double Mellin case would be self-evident.
4.1.3 DGLAP Evolution Formalism

In this subsection, I formulate the solution of the moment space DGLAP equation which is a basics of our analysis. As mentioned before, in momentum space, DGLAP equations, eqs. 3.3 and 3.8 reduces matrix differential equations. Let us simply consider non-singlet case, eq. 3.8 of the type

$$\frac{\partial}{\partial \ln \mu_F^2} q(\mu_F^2) = P(\alpha_s(\mu_R^2)) q(\mu_F^2), \quad (4.17)$$

where I omitted $n$ dependence of $q$ and $P$, and explicitly denoted $\mu_R^2$ dependence of $\alpha_s$ to make its real nature clear. Note that as mentioned before we restrict ourselves in $\mu_R^2 = c \mu_F^2$ with $c$ an arbitrary constant, and we can regard $\mu_R$ as a function of $\mu_F$. Remembering eq. 2.100 we can switch to $\alpha_s(\mu_R^2)$ as the independent variable by

$$\beta(\alpha_s) \equiv \frac{g_s}{4\pi} \beta(g_s) = \frac{\partial}{\partial \ln \mu_F^2} \frac{g_s^2(\mu_R^2)}{4\pi} = \frac{\partial}{\partial \ln \mu_F^2} \alpha_s(\mu_R). \quad (4.18)$$

Then eq. (4.17) becomes

$$\frac{\partial}{\partial \alpha_s} q(\alpha_s) = \beta(\alpha_s)^{-1} P(\alpha_s) q(\alpha_s), \quad (4.19)$$

where I omitted $\mu_R^2$ in $\alpha_s$ and changed variables of distribution from $\mu_F^2$ to $\alpha_s$. When we rewrite $\beta(\alpha_s)$ and $P(\alpha_s)$ in the expansion form as

$$\beta(N, \alpha_s) = - \sum_{k=0}^{\infty} \alpha_s^{k+2} \beta_k(N), \quad (4.20)$$

$$P(N, \alpha_s) = \sum_{k=0}^{\infty} \alpha_s^{k+1} P_k(N), \quad (4.21)$$

then we obtain the following expansion form of eq. (4.19)

$$\frac{\partial}{\partial \alpha_s} q(N, \alpha_s) = - \frac{1}{\alpha_s} \left[ R_0(N) + \sum_{k=1}^{\infty} \alpha_s^k R_k(N) \right] q(N, \alpha_s), \quad (4.22)$$

where

$$R_0 \equiv \frac{1}{\beta_0} P_k \quad R_k \equiv \frac{1}{\beta_0} P_k - \sum_{i=1}^{k} \beta_i P_i. \quad (4.23)$$
This can be easily solved, and we obtain up to NLO level

\[
\frac{q(\alpha_s)}{q(\alpha_s^0)} = \left( \frac{\alpha_s}{\alpha_s^0} \right)^{-R_0} \exp \left( -(\alpha_s - \alpha_s^0) R_1 \right) \quad (4.24)
\]

\[
\Rightarrow q(\mu_F^2) = \left( \frac{\alpha_s(\mu_R^2)}{\alpha_s^0(\mu_R^2)} \right)^{-R_0} \left( 1 - (\alpha_s(\mu_R^2) - \alpha_s^0(\hat{\mu}_R^2)) R_1 \right) q(\hat{\mu}_F^2) \quad (4.25)
\]

where \(\alpha_s^0\) is a value at a initial scale of \(\hat{\mu}_R^2 = c\hat{\mu}_F^2\). In case of singlet evolution, \(R_s\) become matrices. The derivation of the solution becomes more complicated. However it is still possible to entangle the order by order solution [138, 176]. I don’t go into the detail here, instead I list only the results up to NLO. Detailed derivation in arbitrary orders can be found, for example, in [167].

\[
q(\mu_F^2) = \left\{ \left( \frac{\alpha_s}{\hat{\alpha}_s} \right)^{-r_-} e_- - (\alpha_s - \hat{\alpha}_s) e_- R_i e_- \right\} + (\leftrightarrow) \quad (4.26)
\]

\[
+ \left( \alpha_s \left( \frac{\alpha_s}{\hat{\alpha}_s} \right)^{-r_- r_+} - \hat{\alpha}_s \right) \frac{e_- R_i e_+}{r_+ - r_- - 1} \quad (4.27)
\]

where \(q\) is a vector of the singlet distributions of eq. 3.3, and \(R_s\) are defined as eq. 4.23 with a matrix of the singlet splitting kernel in eq. 3.3, \(r_{pm}\) are eigen values of the LO splitting kernel matrix, i.e.,

\[
r_{\pm} = \frac{1}{2\beta_0} \left[ P_{qq,0} + P_{gg,0} \pm \sqrt{(P_{qq,0} - P_{gg,0})^2 + 4P_{gg,0}P_{gq,0}} \right], \quad (4.28)
\]

the matrices \(e_{\pm}\) are the corresponding projector to the eigen vectors, i.e.,

\[
e_{\pm} = \frac{1}{r_{\pm} - r_{\mp}} \left[ R_0 - r_{\pm} I \right] \quad I : \text{unit matrix}, \quad (4.29)
\]

and finally I abbreviated \(\alpha_s(\mu_R^2)\) and \(\alpha_s(\hat{\mu}_R^2)\) as \(\alpha_s\) and \(\hat{\alpha}_s\) respectively.

Applying eqs. 4.25 and 4.27 arbitrary combinations of distributions can be evolved from an initial scale to any required scale. Actually we need the matching conditions for PDFs and the running coupling at a scale where the numbers of active flavors changes by one. The conditions we implemented will be given in section 4.4. This framework was constructed for unpolarized PDFs, longitudinally polarized PDFs (helicity distributions), and (un)polarized fragmentation functions, implementing their appropriate splitting functions. For splitting functions up to NLO, we applied those in [129] for
unpolarized PDFs, [132] for helicity distributions, and [130] for fragmentation functions. (For those of polarized fragmentation function, refer to [120].)

For complement, the short distance cross sections in the moment space are applied from [129, 78] for unpolarized DIS, [141] for longitudinally polarized DIS, [66] for SIA, and [121] for unpolarized and polarized SIDISs. For Drell-Yan process, we keep the rapidity $y$ unintegrated. Then the results cannot be given in the conventional convolution form with PDFs as I mentioned in the previous chapter. We applied $x$ space expressions in [157, 177, 150] to our framework. Along with [121], we can numerically force those $x$ space expressions to convert to $n$ moment space, then we can treat them also in the moment space framework. Because of our current interest of treating it as a reference process, we did not apply the numerical moment space conversion to the current fit analysis although its application would be straightforward as shortly explained in section 4.3.

### 4.2 $\chi^2$ Fit and Statistical Errors

There were several discussions [166, 178, 179, 180] on the treatment of $\chi^2$, i.e., the choice of $\Delta\chi^2$, in polarized distribution community. The conflict seemed from the misleading statistical meaning of the $\chi^2$ fitting. Thus, in this section, I would like to comment on it first. After that, I will define statistical errors and correlations along with our fit.

#### 4.2.1 $\chi^2$ goodness test of fit

The basics of statistics is the idea of maximum likelihood. The principle of maximum likelihood provides a method of how to estimate the true statistical distributions of observables from finite number of observed samples. Let $x_i$ a set of samples $n$ of an observable $x$, and $f(x, a)$ a true but unknown distribution which $x$ obeys with $a$ a set of unknown parameters which characterizes the distribution, like the center value $\mu$ or its variance $\sigma$. We like to estimate the parameters $a$ most efficiently from observables $x_i$. The likelihood $L$ is defined as a probability in which the set of samples are observed;

$$L = f(x_1, a) \times f(x_2, a) \times \cdots f(x_n, a).$$  \hspace{1cm} (4.30)

The principle of maximum likelihood says that the best estimated values of $a$ are determined so that $L$ becomes maximum. Note that the adequacy of the
estimation or equivalently the assumed statistical distribution itself should be checked somehow later.

\[ \frac{\partial}{\partial a_i} L = 0, \quad \text{or} \quad \frac{\partial}{\partial a_i} \ln L = 0. \]  

(4.31)

This principle provides a fundamental tool to find out the best parameters estimated from actual observables, and becomes the basis of usually applied statistical treatments, like taking average for center value, root mean square for standard deviation, and standard deviation of the mean for the error of the center value, et al. In the following, I assume the statistical distribution of observables obeys the normal Gaussian distribution as a limiting distribution.

The principle is applied to a fit to determine some parameters in theory which is expected to describe measured experimental results. Consider a set of \( n \) independent measurement \( y_i \) at known \( x_i \) is assumed to fluctuate statistically by Gaussian distribution with measured statistical error \( \sigma_i \) as it is expected from the measurement. (The true value of \( y_i \) is expected to exist around \( y_i \) with the accuracy of \( \sigma_i \) in sense of Gaussian distribution.) Then we presume a theory \( F(x_i, a_j) \) describing the behavior of \( y_i \) as a function of \( x_i \). In other words, we expect that \( F(x_i, a_j) \) is a true theory and \( F(x_i, a_j) \) surely describes the true value of \( y_i \). Thus we believe that \( y_i \) distributes around \( F(x_i, a_j) \) with the variance \( \sigma_i^2 \). In that case, the true parameter of \( a_j \) is most effectively estimated from the measurement \( y_i \) by the maximum likelihood as before defining

\[ \ln L = \chi^2(a_j) \equiv \sum_i \frac{(y_i - F(x_i, a_j))^2}{\sigma_i^2}. \]  

(4.32)

Note that we took only the power of exponential of Gaussian and removed its minus sign. The best estimated value of \( a_j \) is then determined by the criteria

\[ \frac{\partial}{\partial a_j} \ln L = 0. \]  

(4.33)

In other words, the best parameters are determined by minimizing the quantity \( \ln L \) in eq. 4.32 As well as before, the adequacy of the hypothesis of a theory must be inspected.

For the test of the adequacy, we usually apply a test, named \( \chi^2 \) goodness test of fit. The first notation is that if a set of \( N \) sample \( x_i \) surely obeys random Gaussian distribution of a mean \( \mu \) and variance \( \sigma^2 \) independently,
the quantity
\[ Q = \sum_i \frac{(x_i - \mu)^2}{\sigma^2} \]  
(4.34)
distributes statistically as \( \chi^2 \) distribution, e.g., [181]. The \( \chi^2 \) distribution is expressed by a probability density function of
\[ f(x, k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} , \]  
(4.35)
where \( \Gamma \) denotes the Gamma function and \( k \) is called (the number of) degrees of freedom (d.o.f) defined by the number of statistically independent (random) samples [182, 183]. For example, in the above Gaussian case, the number is defined as \( N \) if \( \mu \) and \( \sigma \) is already known somehow. However we usually estimate them from the samples \( x_i \)s by the method of the maximum likelihood. Then due to the two conditions of eq. 4.31 for them, the number of free statistically random samples is reduced by two, so in this case d.o.f becomes \( N - 2 \). The \( \chi^2 \) distributions with different \( k \) are shown in fig.4.5 for small \( k \).

![Figure 4.5: \( \chi^2 \) distributions for small number of degree of freedom](image-url)

The \( \chi^2 \) distribution has its mean at \( k \) and variance of \( 2k \). Thus \( x_i \) are truly distributing as Gaussian randomly, the quantity \( Q \) in eq. 4.34 is expected to distributes around \( k \) within the accuracy of variance \( 2k \). To be more
quantitative, the confidence level is usually defined as one side integration of the distribution

$$CL(s) = \int_{s}^{\infty} dx f(x, k),$$  \hspace{1cm} (4.36)$$

where \( s \) is some point in \( x > 0 \). Usually \( CL = 0.05 \) is taken as a criteria for judging the nature of randomness of the sample \( x_i \). The confidence curve of low \( k \) is shown in fig. 4.6. As we can see from fig. 4.6, the distribution

![Figure 4.6](image_url)

Figure 4.6: The curves of confidence level of \( \chi^2 \) distribution as a function of degrees of freedom

converges around \( \chi^2 \sim k \) as \( k \) increases. Actually \( \chi^2 \) distribution is known to asymptotically approach to the Gaussian normal distribution with mean \( k \) and variance \( 2k \) as the degree of freedom increases. Thus we can instead estimate the validity of the randomness of the sample roughly by \( k \pm \sqrt{2k} \) with 68.2% confidence if \( k \) is enough large.

If we look back at eq. 4.32 from this basis of \( \chi^2 \) distribution, we notice that the quantity \( \chi^2 \) define in eq. 4.32 can be identified with the statistical variable of \( \chi^2 \) distribution, eq. 4.34 if a theory \( F(x_i, a_j) \) really describes the random Gaussian distribution of measured \( y_i \) around it. Saying other words, we can investigate the validity of the theory by evaluating the value of the \( \chi^2 \) after it is optimized by its minimization. If the theory is expected to be true, the value of the \( \chi^2 \) would reduce to roughly around its degrees of freedom within its statistical fluctuation. As we can see from the above
discussion, this time the degree of freedom $k$ becomes $k = n - m$, where $m$ is the number of parameters determined by the minimization. (In some literature, the value $n - m - 1$ is used for the degrees of freedom. I looked up several references, but I have to confess that I could not find any clear reason to choose that one unless there is a normalization constraint on the theoretical distribution unlike the case of our interest. Note that in any case where its actual size increased this difference is marginal.) Then the (statistical) value $\chi^2$ is expected to be around $k$ within the range of root $2k$ with 68.2% accuracy. We usually define the reduced $\chi^2$, $\tilde{\chi}^2$, to see the consistency clearer, which is defined as

$$\tilde{\chi}^2 = \frac{\chi^2}{k}. \quad (4.37)$$

Then the reduced $\chi^2$ would be around unity with the standard accuracy of $\sqrt{2/k}$. (This $\chi^2$ test procedure is a basis of the well known linear regression.)

As a final note, in our perturbative QCD fit, we apply much more complicated function of parameters so that we cannot solve the minimum conditions analytically unlike the polynomial regressions. To find out the minimum I adopted CERN library based algorithm MINUIT [184].

For the fit of the helicity distributions, we applied the following simple form of $\chi^2$ with the assumption that (uncorrelated) systematic error has the same behavior as statistical error. It is treated as another source of error (indeed there is no need at all to behave so) following the convention in helicity distribution market, e.g. [132] [63] [65].

$$\chi^2(a_j) \equiv \sum_{i=1}^{n} \frac{(y_i - F(x_i, a_j))^2}{\sigma_{i,\text{stat}}^2 + \sigma_{i,\text{syst}}^2}, \quad (4.38)$$

where $\sigma_{\text{stat}}$ is the statistical error and $\sigma_{\text{syst}}$ the (uncorrelated) systematic error.

For the fit of the fragmentation functions, several experiments gave the scale systematic uncertainties for their results in addition to usual statistical and systematic errors. Those uncertainties cannot be treated as independent on each kinematic point. Thus this should be handled as correlated error. To deal with these correlated errors, we applied the following $\chi^2$ form;

$$\chi^2(a_j, s_l) \equiv \sum_{i} \sum_{l} \frac{(y_i - F'(x_i, a_j))^2}{\sigma_{i,\text{stat}}^2 + \sigma_{i,\text{syst}}^2} + \sum_{l} s_l^2$$

$$F'(x_i, a_j) \equiv F(x_i, a_j)(1.0 + s_l \Delta_l), \quad (4.40)$$
where $\sum_l$ runs over various experiments ($N_E$) and $\sum_i$ over experimental data ($N_l$) given from each experiment. $\Delta_l$ is the (absolute) scaling uncertainty given from each experiment and $s_l$ is a new parameter which controls the contribution of the uncertainty to its experimental data and is optimized by fit. As it is clear from the form of the $\chi^2$ of eq. (4.40), we assume the uncorrelated error obeys Gaussian distribution as well with the standard deviation $\Delta_l$. This is achieved in eq. (4.40) through $s_l$ which has a zero mean and a unit standard deviation, i.e., $\sum_l N_E (s_l - 0.0)^2 / 1.0$.

For more general treatment of the correlated systematic errors, we can refer to several studies in unpolarized PDF market, e.g., [42, 41]. The efficient handling of those errors with uncorrelated errors in $\chi^2$ fit is proposed in [42].

### 4.2.2 Error estimation in $\chi^2$ fit

The next issue related to the $\chi^2$ fit is to estimate how well the parameters $a_j$ in eq. (4.32) are determined in the fit. If we obtain another set of experimental samples, the determined $a_j$ values would vary statistically set by set. We would like to estimate its statistical ambiguities related to the $\chi^2$ minimization. As in case of the error estimate as $\sigma / \sqrt{N}$ of the mean value of the samples normally distributed, the information on the statistical error of the determined parameters would exist in the power of the exponential, i.e. $\chi^2$ itself. Let $\chi^2_0$ the minimum of the $\chi^2$ defined in eq. (4.32) and $a_{0,j}$ the optimized parameters evaluated at the minimum. Noting that the $\chi^2$ is a function of parameters $a_j$, we can expand the $\chi^2$ around the minimum for $\hat{a}_j = a_j - a_{0,j}$ as

$$
\Delta \chi^2 = \chi^2 - \chi^2_0 = \hat{a}_i \cdot \nabla_i \chi^2 |_{a_j = a_{0,j}} + \frac{1}{2!} (\hat{a}_i \cdot \nabla_i)^2 \chi^2 |_{a_j = a_{0,j}} + O((\hat{a}_i)^2) \quad (4.41)
$$

$$
\cong \hat{a}_i \hat{a}_j \frac{1}{2} \frac{\partial^2 \chi^2}{\partial a_i \partial a_j} |_{a_j = a_{0,j}}, \quad (4.42)
$$

where $\nabla_i$ denotes partial differentiate on $a_i$. Then the first term on the r.h.s of eq. (4.41) drops by the minimum conditions. We assume the higher order contributions, $O$, are negligible in neighborhood of the minimum. Then we see from eq. (4.42) that the statistical Gaussian distributions of the measures $y_i$ around $F(x_i, a_j)$ can be regarded as those of $a_j$ around $a_{0,j}$ with the covariant matrix $U_{i,j}$ of

$$
U_{i,j}^{-1} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial a_i \partial a_j} |_{a_j = a_{0,j}}. \quad (4.43)
$$

(We can directly check its role as a covariant matrix by taking statistical moments $E$ of $a_i$, e.g., [183].) More clearly, the symmetric covariance matrix
$U_{i,j}$ is expressed in terms of the standard variance $\sigma_i^2 = E[(a_j - a_{0,j})^2]$ and covariance $\text{cov}(\hat{a}_i, \hat{a}_j) = E[(a_j - a_{0,j})(a_i - a_{0,i})]$ as

$$U_{i,j} = \begin{pmatrix} \sigma_1^2 & \text{cov}(\hat{a}_1, \hat{a}_2) & \text{cov}(\hat{a}_1, \hat{a}_3) & \cdots \\ \text{cov}(\hat{a}_2, \hat{a}_1) & \sigma_2^2 & \text{cov}(\hat{a}_2, \hat{a}_3) & \cdots \\ \text{cov}(\hat{a}_3, \hat{a}_1) & \text{cov}(\hat{a}_3, \hat{a}_2) & \sigma_3^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$  \hfill (4.44)

Thus we can investigate the statistical fluctuations of determined $a_j$ along with those of $y_i$ through the covariant matrix $U_{i,j}$ which determines the behavior of $\chi^2$ as deviation from its local minimum. As $\Delta \chi^2 > 0$, the $\chi^2$ shapes multidimensional quadratic curve, fig. 4.7 in the neighborhood of the local minimum unless the covariant matrix is singular. Additionally, from eq. 4.42 we can see that we can obtain the components of the covariant matrix through the cross section of the quadratic curve at a $\Delta \chi^2 = 1$ plane, fig. 4.8.

![Figure 4.7: $\chi^2$ curve in the parameter space near the local minimum](image)

Once we obtained the variances or covariances of parameters $a_j$, we can ask the usual error propagation method to evaluate the statistical error or correlation of some quantities, say $A(a_j), B(a_j)$, described by the parameters. By the linear approximation of $A(a_j), B(a_j)$ around $a_{0,j}$, we can obtain

$$\sigma_A^2 = U_{i,j} \frac{\partial A}{\partial a_i} \frac{\partial A}{\partial a_j}, \quad \sigma_B^2 = U_{i,j} \frac{\partial B}{\partial a_i} \frac{\partial B}{\partial a_j},$$  \hfill (4.45)

$$\text{cov}(A, B) = U_{i,j} \frac{\partial A}{\partial a_i} \frac{\partial B}{\partial a_j}.$$  \hfill (4.46)
where all the partial derivatives are taken at $a_j = a_{0,j}$. A theory which provides a singular covariant matrix is thus meaningless in usual sense of the statistical error estimation, and the usual treatment of the statistical errors breaks down. There are several ways to be applied for the estimation of errors of fit, e.g., [64, 65]. However we would like to keep along with this definition of statistical errors with the parabolic nature of $\chi^2$ curve as long as possible because of its clear picture of error and ease of handling for qualitative error estimation of a quantity. One note related to the $\chi^2$ curve in multi-parameter space is that the confidence level of the region bounded by $1 \sigma_j$ of parameters in the multi-parameter space, of course, decreases as the number parameter increases [13]. In papers [178, 179], the authors took $\Delta \chi^2$ value which corresponds to keeping 68.2% confidence level region in the parameter space. It makes the applied $\Delta \chi^2$ to the error estimation significantly increase. ($\Delta \chi^2 \sim$ the number of parameters.) However, in sense of the error of a parameter with the standard deviation, we would still ought to keep $\Delta \chi^2 = 1$.

In the MINUIT, the search of $\Delta \chi^2 = 1$ multi-dimensional surface and eigen vectors of the multi-dimensional quadrature can be performed by a method named HESSIAN implemented in the MINUIT package. Note, however, in several occasions [64, 42, 43, 41], the ideal condition of the parabolic assumption breaks down because of the lack of enough efficient data to fix all
the parameters. The redundancy leads to a very flat direction in the eigenvalue space (a very large or small eigenvalue of the covariance matrix) which means that cubic, quartic, etc. terms dominate. During the process of diagonalization in \textit{MINUIT} this bad behavior feeds through into the whole set of eigen vectors to a certain extent. Thus, in order that the Hessian method work as well we have to remove the redundancy from input parameters. In order to do this, we simply fix some of the parameters at their best fit values so that the covariant matrix only depends on a subset of parameters which are sufficiently independent. Then the quadratic approximation becomes reasonable. Noting that the flatness of the parameter space means much less influence on the theoretical calculation, it would be justified to fix them at the best local minimum.

Final remark is that, based on several cases, the value of the $\Delta \chi^2$ is chosen as much larger value like $\Delta \chi^2 = 100 \[42\]$ or $\Delta \chi^2 = 50 \[43\]$. Basic reason for those choices is that they contains large amount (order of thousands) of experimental data from different sources. Some of data sets can be unacceptably poor in sense of pQCD prediction due to the potential theoretical uncertainty, model dependent error or non-Gaussian distribution. They have considered that all of the current data sets must be acceptable and compatible at some level. To compensate unsatisfactory value of reduced $\chi^2$ compared with the number of degrees of freedom, they enlarged the range of $\Delta \chi^2$ and tried to absorb the inadequacy into the error of parameters. In our analysis, we also obtained rather large reduce $\chi^2$ values in sense of the degrees of freedom. We would have to consider the origin of deviation much carefully, and, if needed, properly estimate the adequate errors with cautious treatment of systematic errors.

### 4.2.3 Lagrange multiplier method

There is a practical method to inspect the $\chi^2$ behavior along with a deviation of a quantity, say $A(a_j)$, from optimized value with $a_{0,j}$. From the $\chi^2$ behavior we can catch up more instinct picture of the statistical error of $A(a_j)$. This is called Lagrange multiplier method \[185\]. In the Lagrange multiplier method, we define a modified $\chi^2$, $\Psi$, as

$$\Psi(a_i, \lambda) = \chi^2(a_i) + \lambda A(a_i). \quad (4.47)$$

First we set $\lambda$ at a value. Then try to minimize $\Psi$ as a function $a_j$. Using the new set of $a_j$, $a_{\lambda,j} \neq a_{0,j}$, we can calculate $\chi^2(a_{\lambda,j})$ and $A(a_{\lambda,j})$. The given $A(a_{\lambda,j})$ provides the minimum $\chi^2$ among other sets of parameters which gives
the same $A$ value, fig. 4.9. Iterating this process, we can obtain $\chi^2$ curve as a function $A$, fig. 4.9. To recognize the concrete meaning of this curve and its connection to the error, I would like to show an analytical result of this problem. In the following I’m going to consider only the deviated values of quantities from those given at the local minimum. Thus I’m going to omit the accent symbol $\hat{}$ to indicate the deviation everywhere in the following. We assume we obtained the positive definite covariant matrix at the minimum. Now consider the problem to find the minimum $\chi^2$ in the condition that $A^2(a_j) = \text{constant } (c)$, i.e., investigate the extremum of the following quantity.

$$\psi(a_i, \lambda) = \Delta \chi^2(a_i) + \lambda(A^2(a_i) - c).$$

(4.48)

Along with the usual Lagrange multiplier method, the minimization of $\psi$ is achieved by the following conditions;

$$\partial \psi(a_i, \lambda)/\partial a_i = 0, \quad \partial \psi(a_i, \lambda)/\partial \lambda = 0.$$  

(4.49)

Remembering eqs. 4.42 and 4.43, the conditions can be rewritten as

$$U_{ij}^{-1}a_j + \lambda A \partial A/\partial a_i = 0,$$  

(4.50)

$$A^2(a_i) = c.$$  

(4.51)

where $U_{ij}$ is the symmetric covariant matrix of eq. 4.44. From the fact $U_{ij}U_{jk}^{-1} = \delta_{ik}$, the first condition becomes

$$a_i = -\lambda AU_{ij} \partial A/\partial a_j.$$  

(4.52)
Then the $\Delta \chi^2$ deduces to the following form;

$$
\Delta \chi^2 = a_i U^{-1}_{i,j} a_j = (\lambda A)^2 a_k U^T_{k,i} U^{-1}_{i,j} a_l a_l
$$

(4.53)

$$
= (\lambda A)^2 \frac{\partial A}{\partial a_i} U_{i,j} \frac{\partial A}{\partial a_j}.
$$

(4.54)

From eq. 4.52, we can also obtain

$$
a_i \frac{\partial A}{\partial a_i} = \lambda A \frac{\partial A}{\partial a_i} U_{i,j} \frac{\partial A}{\partial a_j}.
$$

(4.55)

Thus we obtain

$$
\Delta \chi^2 = \left( a_i \frac{\partial A}{\partial a_i} \right)^2 \left/ \left( \frac{\partial A}{\partial a_i} U_{i,j} \frac{\partial A}{\partial a_j} \right) \right.
$$

(4.56)

When we note eq. 4.45 and the fact that we can take

$$
\hat{A} \simeq \hat{a}_i \frac{\partial \hat{A}}{\partial \hat{a}_i} |_{\hat{a}_j = 0} = \hat{a}_i \frac{\partial A}{\partial a_i} |_{a_j = a_{0,j},}
$$

(4.57)

around the minimum (as the error propagation), then we see that eq. 4.56 becomes the following obvious closed form on $\hat{A}^2$.

$$
\Delta \chi^2 (\hat{A}^2) = \frac{\hat{A}^2}{\sigma^2_A}.
$$

(4.58)

Thus, from eq. 4.51 $\Delta \chi^2$ curve which is deduced from the minimization of 4.48 or equivalently 4.47 becomes only a function of $A$ and shapes parabolic form, fig 4.9. We can also say that the $\chi^2$ curve given from the Lagrange multiplier method is a projection of $\chi^2$ distribution in the multi-parameter space onto $\chi^2$-$A$ plane. (Note that all the correlations have already been taken into account as it is clear from the above derivation.) In that curve, $\Delta \chi^2 = 1$ surely corresponds to the (propagated) error of $\sigma_A$ in that plane, fig 4.9 as it is clear from eq. 4.58. The Lagrange multiplier method provides the $\chi^2$ behavior of a quantity more pictorially.

### 4.3 Conceptual Chart of Fit Process

In this section, I'm going to briefly review our fitting procedure in the calculation framework which we newly constructed with the Mellin transform technique. Fig. 4.10 shows the conceptual calculation flow in our framework. We constructed our framework in C++ language because of the wide
spreading programing to achieve the fit procedure with various high energy processes. The structural concept of C++ fits well our purpose of handling many processes based on general theoretical background of perturbative QCD (pQCD). By virtue of it, we could construct a very general framework for perturbative QCD calculations. It also greatly helped us for constant improvement and maintenance. In the following, I’m going to shortly comment on each of four parts in fig. 4.10 in order.

In the first part, shown as ① in fig. 4.10 we set initial distributions of partons at an initial scale with assumed functional forms. Widely used functional forms in pQCD fit market are those which can be generally expressed as

\[ f(x, \mu_0^2, F) = \sum_{i,j=0} a_{ij} x^{\alpha_i} (1-x)^{\beta_j}, \]  

(4.59)

where \(a_{ij}, \alpha_i, \beta_j\) are some parameters which would be determined by fit procedure, and \(\mu_0, F\) is an initial energy scale which is usually taken at 1 GeV in our fit. The greatest advantage of those functional forms is that those can be analytically converted into Mellin space with the beta function \(B(n, m)\) defined as

\[ B(n, m) = \int_0^1 dt t^{n-1} (1-t)^{m-1} = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}, \]  

(4.60)

where \(\Gamma\) is the gamma function and \(n, m\) are complex variables. You can easily see that the functions of the form of eq. 4.59 are expressed as the sum of the beta functions when the Mellin transform of eq. 4.1 is taken. These simplified Mellin transformed forms helps us to perform fast fit routine with the consistent calculation manner. However, some groups like 42 introduced much complex functional forms which cannot be analytically converted. To make these functions also calculable in our framework, we additionally implemented a routine which numerically convert a x space function into Mellin space. Thus we can, in principle, perform a fit with the initial distributions of arbitrary functional forms in our fit. However we still stick to the functional forms of eq. 4.59 because our fit for the helicity distributions and fragmentation functions does not effectively require more complicated functional forms. Actually, even in unpolarized PDF market, the functions of CTEQ group, e.g., 42, 185, and those of MRST group e.g., 61, 60 who introduces the traditional form of eq. 4.59 concedes well consistent results in sense of several effective predictions. The actual functional forms of initial distributions used in our fit are reviewed in section 4.4.
In the second part, shown as ② in fig. 4.10, the initial distributions are evolved from the initial scale $\mu_0, F$ to an arbitrary scale $\mu_F$ by DGLAP equations with the appropriate splitting functions as was explained in 3.1. When the initial distributions are converted into the Mellin space, the equations simply reduces to homogeneous first-order matrix differential equations on the energy scale. Thus, after numerically solving the evolution equation, we can obtain an arbitrary complex value of the distributions at an arbitrary scale. Its actual calculation was given in subsection 4.1.3. The splitting functions applied in our Mellin space calculation were already indicated in the same subsection.

In those evolutions, there is a subtlety for the treatment of heavy quarks which has masses heavier than the initial scale $\sim 1$ GeV, i.e. charm, bottom, top. The treatment of those masses in our framework is explained in section 4.5.

In the third part, ③ in fig. 4.10 the evolved distributions are combined with the coefficient functions, i.e. $C$ in fig. 4.10 to build up appropriate observables like structure functions, asymmetries, etc. Note that those observables are still in the Mellin space. The observables we are going to handle were listed in chapter 3. Those adopted coefficient functions converted to Mellin space were listed in subsection 4.1.3.

There are processes which cannot be analytically transformed into Mellin space. Those processes in general handle angle dependence like rapidity dependent differential cross section of Drell-Yan process. In our scope of analysis, those processes are not included in our fit. Thus we put aside the application of numerical conversion of those coefficient functions into Mellin space. As a short remark, this conversion can be achieved by the analogous fashion to the numerical conversion of the initial distribution with an arbitrary functional form. The biggest difference is that we can prepare some map of the complex values of the coefficient functions in the Mellin space beforehand because the coefficient functions are independent of the parton distributions, i.e., those do not vary in fit routine. The actual application of this procedure to our fit analysis is kept for a future subject.

There is again a subtlety related to the mass of heavy quarks in the calculation of the observables. Our treatment of those masses is also explained in section 4.5.

Finally in the forth part, ④ in fig. 4.10 we take inverse Mellin transform of the observables given in the previous process to compare them to actual experimental data given in $x$ space. The Mellin inversions and appropriate moments if necessary are calculated following the manner explained in section
The data set used in our fit will be listed in section 4.6. The deviation of the experimental data and corresponding calculated quantities are minimized by $\chi^2$ fitting procedure with MINUIT algorithm which was explained in section 4.2. The optimization yields best values of parameters of initial distributions which result in the optimum description of parton composition of hadrons. By virtue of the Mellin transform method, the computation time for one loop routine in our fit became considerably shorter than our earlier framework of direct $x$ space calculation. To be more quantitative, in the next leading order calculation, it became shorter by a factor of thousand. (Of course it depends on how accurate those calculations are.)

As the final remark, the Mellin inversion can be of course applied at parton distribution level. Besides of the construction of the observables within the moment space, we also implemented $x$ space integration for the construction, i.e., take inverse Mellin transform at parton level and then perform the convolution integral directly between the distribution and $x$ space coefficient functions. This additional preparation of $x$ space coefficient functions was used as consistency check for our Mellin space calculation. This $x$ space integration framework also becomes fundamental to the numerical Mellin conversion of the coefficient functions of the processes which are impossible to be analytically converted.
Figure 4.10: conceptual chart of our calculation framework for global QCD analysis
4.4 Initial Distributions and Constraints

In this section, I will introduce the initial distributions defined in principle at $\mu^2_{F,0} = 1 \text{ GeV}^2$ and several constraints imposed on these initial distributions. The first subsection is devoted for fragmentation functions and The second is for the helicity distributions.

4.4.1 Fragmentation Functions

Here I will indicate the constraints set for the initial distributions of the fragmentation functions denoted $D^H_{q,g}(z, \mu^2_{F,0})$.

As mentioned in section 3.3, I’m going to assume the charge conjugation symmetry of eq. 3.56.

$$D^H_q(z, \mu^2_F) = D^R_{\bar{q}}(z, \mu^2_F).$$  \hspace{1cm} (4.61)

As the functional form of the following fragmentation functions, I assumed, for every fragmentation function, the form

$$D^H_{q,g}(z, \mu^2_F) = \eta \times z^\alpha (1-z)^\beta / B(\alpha + 2, \beta + 1)$$  \hspace{1cm} (4.62)

with fit parameters $\eta$, $\alpha$ and $\beta$, where the beta function $B(n, m)$ was installed (eq. 4.60) to make the parameter $\eta$ the second moment of the fragmentation function. As it will be discussed in the next section, the heavy quarks, charm and bottom quarks in my analysis, were turned on at each of the mass threshold. I still took the same functional form for these distributions. Note also that in principle there is no constraint on gluon fragmentation function.

**Pion Fragmentation Function**

For the fragmentation to $\pi^+$, I set the constraints based on flavor $SU(2)$ symmetry between up and down quarks;

$$D^\pi_{u^+}(z) = D^\pi_{d^+}(z)$$ \hspace{1cm} (4.63)

$$D^\pi_{d^+}(z) = D^\pi_{u^+}(z).$$ \hspace{1cm} (4.64)

The first one is occasionally called *favored* fragmentation function which should dominate the fragmentation process, and the second as *dis-favored* fragmentation function. For other distributions, I also assumed

$$D^\pi_{s^+}(z) = D^\pi_{s^+}(z), \quad D^\pi_{c^+}(z) = D^\pi_{c^+}(z), \quad D^\pi_{b^+}(z) = D^\pi_{b^+}(z),$$ \hspace{1cm} (4.65)
expecting the appearance of these contributions through the process of pair creation $g \rightarrow q\bar{q}$.

These conditions are kept unchanged in the DGLAP evolution which is insensitive to flavor difference. In my analysis, I did not assume $SU(3)$ symmetry and the strange quark distributions are separated from the disfavored ones, expecting the appearance of some trace of the suppression of $g \rightarrow s\bar{s}$ process in the hadronization. Handling only SIA process, the separation is meaningless because the short distance cross section of the process is insensitive to the difference between down and strange quarks. Because I included hadron multiplicity in my fit, I expected the separation becomes possible through the difference of (unpolarized) parton distributions.

In case of handling neutral pion fragmentation, I assumed

$$D_{q,g}^{\pi 0} = (D_{q,g}^{\pi +} + D_{q,g}^{\pi -})/2.0,$$  \hfill (4.66)

based on the flavor $SU(2)$ symmetry.

**Kaon Fragmentation Function**

For $K^+$ fragmentation functions, I set the following constraints;

$$D_{u}^{K^+}(z) \neq D_{s}^{K^+}(z)$$  \hfill (4.67)

$$D_{d}^{K^+}(z) = D_{u}^{K^+}(z) = D_{s}^{K^+}(z) = D_{d}^{K^+}(z),$$  \hfill (4.68)

corresponding to favored and dis-favored quarks, and also for heavy quarks as

$$D_{c}^{K^+}(z) = D_{c}^{K^+}(z), \quad D_{b}^{K^+}(z) = D_{b}^{K^+}(z).$$  \hfill (4.69)

Again, expecting the breaking of flavor $SU(3)$, I set $D_{u}^{K^+}(z) \neq D_{s}^{K^+}(z)$ for favored fragmentation functions. It is expected that $s \rightarrow K^+$ transition should happen more frequently than $u \rightarrow K^+$ because less energy is needed for the creation of a $u\bar{u}$ pair from the vacuum than for a $s\bar{s}$ pair.

**Proton Fragmentation Function**

For fragmentation functions to proton, I set

$$D_{u}^{p^+}(z) = 2D_{d}^{p^+}(z),$$  \hfill (4.70)

$$D_{u}^{p^+}(z) = D_{d}^{p^+}(z) = D_{s}^{p^+}(z) = D_{s}^{p^+}(z),$$  \hfill (4.71)
corresponding to favored and dis-favored fragmentation functions, and again for heavy quarks as
\[ D_p^+(z) = D_c^+(z), \quad D_b^+(z) = D_b^+(z). \] (4.72)

I neglected the subtlety of \( SU(3) \) breaking for proton case because the presently available experimental information for fragmentation to proton is not rich enough to resolve such difference. (There are no available multiplicity data for proton neither.)

**Fragmentation Function for Charged Hadrons**

For the fragmentation functions for the charged hadrons, I took the sum of the above three fragmentation functions because there would be no other long living charged hadrons, \( c_\tau \sim O(m) \), which are detected as final state charged hadrons in detectors.
\[ D_{h,q,g} = D_{\pi,q,g} + D_{K,q,g} + D_{p,q,g}. \] (4.73)

### 4.4.2 Parton Helicity Distributions

In this subsection I will show the constraints for the proton parton helicity distributions \( \Delta q(x, \mu_F^2, 0) \) or \( \Delta g(x, \mu_F^2, 0) \) at the initial scale \( \mu_F^2 = 1 \text{ GeV}^2 \).

First, I constrained the number of active flavors \( n_f \) at \( n_f = 3 \) over the kinematic range of treated experimental data because the contribution from heavy quarks is expected to be marginal for current polarized experimental data.

As the functional form for the helicity distributions, I took
\[ x \Delta q(x) = \eta \times x^\alpha (1 - x)^\beta \times (1 + \delta \sqrt{x})/N \] (4.74)
for \( \Delta u \) and \( \Delta d \) with \( \eta, \alpha, \beta \) and \( \delta \) fit parameters, and
\[ x \Delta q, g(x) = \eta \times x^\alpha (1 - x)^\beta /N', \] (4.75)
for the others with \( \eta, \alpha \) and \( \beta \) fit parameters, where \( N \) and \( N' \) are appropriate sum of the beta functions make the parameter \( \eta \) the first moment of each of the helicity distributions.

I treated the data with polarized proton, neutron and deuteron target or beam. Assuming the flavor \( SU(2) \) symmetry, the neutron distributions were
set to be equal to those of proton except the interchange between up and down flavors, i.e.,

\[
\Delta u_p = \Delta d_n, \quad \Delta d_p = \Delta u_n, \\
\Delta \bar{u}_p = \Delta \bar{d}_n, \quad \Delta \bar{d}_p = \Delta \bar{u}_n.
\]

(4.76) \hspace{1cm} (4.77)

While, the deuteron distributions were assumed to be handled simply as an average of neutron and proton distributions, i.e.,

\[
\Delta \{g, g\}_d = \left( \Delta \{g, g\}_p + \Delta \{g, g\}_n \right) / 2.0.
\]

(4.78)

The same manipulation was taken for the calculations of unpolarized quantity.

However, for polarized case, there is a subtlety in the handling of deuteron distributions related to the fact that deuteron is a combined state of proton and neutron. The spin 1 of the deuteron has some contributions from the angular momentum of the proton-neutron system. In other words, it does not exactly mean proton +1/2 and neutron +1/2 states in the deuteron +1, as expected from the naive prescription eq. (4.78). The effect was investigated through meson exchange model \[187, 188, 189, 190, 191\]. In our analysis, we set a factor

\[
1 - \frac{3}{2} \omega_D \quad (\omega_D = 0.058)
\]

(4.79)

following them, and multiplied to polarized deuteron quantities, i.e., \(g_1, g_1^h\) to reflect this effect.

**positivity condition**

As mentioned in section 2.4, naive definition of helicity distributions must satisfy the positivity condition \[2.139\]

\[
|\Delta q(x)| \leq q(x),
\]

(4.80)

based on the probability interpretation. Note that it was given for the bare densities. At the leading order (LO), however, we can see that the condition is kept unchanged for normalized distributions \[192\] considering the obvious positivity of cross sections between polarized and unpolarized processes. It guarantees the probability interpretation for the given distributions at LO. However, beyond the LO, the simple interpretation in general breaks down \[193\], and the condition should be correctly replaced by the positivity conditions of the cross section level. However, as shown in \[193\], the change does
not seem to be large even in the gluon case where the condition becomes most effective. Thus in my analysis, I set the positivity conditions of eq. \ref{eq:2.139} at the initial scale even in the NLO analysis for simplicity. The DGLAP evolution does not violate the condition. This means in turn the condition becomes more strict as the scale where it is imposed decreases. I set it at the edge scale 1 GeV$^2$ where the parton description barely works.

\section*{Constraints for the First Moments}

I set constraints for the first moments of the helicity distributions on the footing of the fermion helicity conservations eq. \ref{eq:3.36}. This corresponds to the number conservation constraints of eq. \ref{eq:3.26} set for proton unpolarized PDFs, i.e., $u(2, \mu^2_F) - \bar{u}(2, \mu^2_F) = 2$ and $d(2, \mu^2_F) - \bar{d}(2, \mu^2_F) = 1$.

The constraints are given from the experimental measurement of the semileptonic decays of the spin $1/2$ hyperons, which belong to octet baryon states. Let us define $B_j$ labelling \cite{194} for the hyperons which is sometimes expressed as $\psi_j$ in some textbooks with $j$ SU(3) octet index, e.g., proton $p(n)$ is expressed as $(B_4(6) + iB_5(7))/\sqrt{2}$. Then the hadronic transition in the decay is in general controlled by the matrix elements of the form $\langle B_i|h^\mu_j|B_j \rangle$ where $h^\mu_i$ is a sum of the possible charged hadronic currents that couples to the $W$ boson in the electro-weak Lagrangian, which can be re-expressed by color octet vector and axial current defined respectively as

\begin{align}
J^\mu_j &= \bar{\psi}\gamma^\mu \left( \frac{\lambda_j}{2} \right) \psi \,(4.81) \\
J^5_{5\mu} &= \bar{\psi}\gamma^\mu \gamma^5 \left( \frac{\lambda_j}{2} \right) \psi \,(4.82)
\end{align}

where $\psi = (u, d, s)$, and $\lambda_j$ is the fundamental representation of corresponding SU(3) generator. For example,

\begin{equation}
\bar{u}\gamma^\mu(1 - \gamma_5)s = J^4_\mu + iJ^5_\mu - (J^4_{5\mu} + iJ^5_{5\mu}), \quad (4.83)
\end{equation}

which describes the transition $\Lambda \rightarrow p$.

Under the assumption of the strict flavor SU(3) with massless quarks (SU(3)$_R \times$ SU(3)$_L$) and neglecting momentum transfer and mass difference between initial state and final state hyperons, all the hyperon semileptonic decays are described in terms of two constants, $F$ and $D$ which occur in the forward matrix elements of the octet axial currents \cite{194} as;

\begin{equation}
\langle B_j; P, S|J_{5\mu}|B_k; P, S \rangle = 2MS_{\mu}(-if_{ijk}F + d_{ijk}D), \quad (4.84)
\end{equation}

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where $S^\mu$ is the spin axial vector, $M$ is the mass of the hyperon, and $f_{ijk}$ and $d_{ijk}$ are the usual $SU(3)$ group constants. It can be derived by the simple current algebra for the $SU(3)$ group. The structure of eq. [4.84] is an example of the Wigner-Eckart theorem. Thus through the hyperon beta decays, we can extract the axial current constituents extrapolating zero momentum transfer limit, and then we can obtain $F$ and $D$ parameters combining several decays. Note that eq. [4.84] is strongly based on the flavor $SU(3)$ symmetry. In nature, it is known the $SU(3)$ symmetries (both vectorial and axial $SU(3)$) are broken by the heavier mass of strange quark while $SU(2)$ symmetries between up and down flavors are well accepted because of the marginal masses of them.

The important notation is that $J_{3\mu}$ defined in eq. has flavor diagonal element of

$$J_{3\mu}^3 = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d,$$

$$J_{3\mu}^8 = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s.\quad(4.85)$$

Remembering the discussion in subsection 2.5, we can interpret the quark local operator for polarized case like eq. [3.31] sandwiched with proton state as the appropriate combination of the $N$th moments of the helicity distributions according to the value of $N$ (eq. [2.156]). Thus noting eq. [2.156] we see that

$$\langle p|J_{3\mu}^3|p\rangle = \Delta u + \Delta \bar{u} - (\Delta d + \Delta \bar{d})\quad(4.87)$$

$$\langle p|J_{3\mu}^8|p\rangle = \Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} - 2(\Delta s + \Delta \bar{s}).\quad(4.88)$$

Thus, remembering $|p\rangle = (|B_4\rangle + i|B_{5(7)}\rangle)/\sqrt{2}$, if we assume the $SU(3)$ vectorial and axial symmetries and that eq. [4.84] holds, we can easily obtain

$$\langle p|J_{3\mu}^3|p\rangle = F + D\quad(4.89)$$

$$\langle p|J_{3\mu}^8|p\rangle = \frac{1}{\sqrt{3}}(3F - D)\quad(4.90)$$

using the concrete values for $f_{ijk}$ and $d_{ijk}$. (More detailed discussion can be found in [195, 196, 197].)

Thus, I took the following constraints for the first moments at the initial scale $\mu_F^2 = 1$ GeV$^2$ using the experimentally extracted $F$ and $D$ parameters;

$$\Delta U - \Delta D = 1.2573 \times (1.0 + \epsilon_{SU(2)}),\quad(4.91)$$

$$\Delta U + \Delta D - 2\Delta S = 0.575 \times (1.0 + \epsilon_{SU(3)}),\quad(4.92)$$

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where $\Delta Q = \Delta q(1) + \Delta \bar{q}(1)$ and $\epsilon_{SU(2)}$ and $\epsilon_{SU(3)}$ are parameters introduced to loosen the assumption of the flavor $SU(2)$ and $SU(3)$ symmetry needed for the derivation of these values. In my analysis I set $\epsilon_{SU(2)} = 0$ while made $\epsilon_{SU(3)}$ a fit parameter. The values are respectively extracted from the neutron $\beta$ decay result, $0.575 \pm 0.016$ [13] ($= g_A/g_V$ in Cabibbo theory [198]), and the hyperon $\beta$ decay results, $1.2573 \pm 0.0028$ [195]. The reason why I did not apply the hyperon decay result for the $SU(2)$ part is that the extraction method is always accompanied by the ambiguity of broken $SU(3)$ and the neutron decay data directly provides it only with the favourable $SU(2)$ symmetry assumption.

Final note is that in the above derivation, we have to invoke the flavor $SU(3)$ symmetry somehow to extract the required values. There are some other approaches to obtain the first moment of helicity distribution using neutrino nucleon scattering trying to extract other combinations of the axial current. For example, The experiments referred in [199, 200, 201] (would) provide us the combination of $\Delta U - \Delta D - \Delta S$ through neutrino-nucleon elastic scattering. Then using only the promising relation eq. [4.91] based on $SU(2)$ symmetry, we (will) obtain $\Delta S$ directly.

4.4.3 unpolarized parton distribution function

For unpolarized PDFs of proton, I basically used MRST 2001 [60]. The initial distributions of their analysis have the simple form of eq. 4.59. Thus I simply applied the initial parameters in my framework and evolved them with the unpolarized splitting functions within my calculation algorithm. I surely checked the consistency of my evolution calculation with their results provided in [202]. For their evolution I always assume $\alpha_s(M_Z) = 0.1175$ following their analysis, where $M_Z \sim 91$ GeV is the mass scale of Z boson resonance. The treatment of the distributions of deuteron and neutron is the same as that of the helicity distributions.

4.5 Treatment of Heavy Quarks and Matching Conditions

Here in this section, I’m going to introduce several general conditions in my fit procedure.

First, for the running coupling constant, equivalently $\alpha_s$, we need a reference value usually taken at the Z boson resonance scale $M_Z = 91.1876$.
GeV^2. More properly this parameter should be treated as a fitted parameter. In my study, however, I kept it a constant for simplicity at the world average value extracted from various processes \cite{13}, i.e., \( \alpha_s(M_Z) = 0.1176 \). (The extraction was based on \( \overline{MS} \) scheme which was explained in chapter 2.) This helps us to shorten the computation time. If I move to the value as a free parameter in my fit, the fit procedure is expected to consumes roughly twice longer time. (For the width of Z boson needed for my analysis of single hadron inclusive measurement in high energy \( e^+ e^- \) annihilation (SIA) data, I took again the world average \( \Gamma_Z = 2.4952 \) GeV. For the electromagnetic coupling, if necessary, I took \( \alpha_e = 1/137.03 \) where the scale dependence is safely neglected.)

For the evolution of the running coupling constant, I applied the next leading order (NLO) evolution equation, i.e., up to \( g_s^4 \) order in eq. 2.27. Remember that in the evolution of the running coupling constant, we need a parameter of the number of the active flavors, \( n_f \), through eqs. 2.28 and 2.29. The question is how we should treat the number.

The problem is based on the existence of the heavy quarks which have heavier masses than the initial scale which I took at \( \mu_R \sim 1 \) GeV. When we start the evolution of the running coupling constant from the initial scale, the number of the active flavors are expected to be three corresponding to the light flavors of up, down and strange. This is because the only effects of heavy quark propagators are in loop corrections and of a form that they can be effectively dropped out with the correction of the order of the inverse of the heavy quark mass, like higher twists effects in the operator product expansion. This is known as the decoupling theorem \cite{203,204}. However, when we work in well above a scale of a heavy quark mass \( m_h \), it is the mass that can be neglected. We can treat the quark as massless on the same footing as the light quarks. Thus we have two regimes: when \( \mu_R \gg m_h \), the heavy quark participates fully as a massless quark, and when \( \mu_R \ll m_h \), we can omit effectively the contribution from the heavy quark and work in the effective theory involving only light quarks (with the correction of the order of \( \mu_R/m_h \)). To cover all the scale ranges, some method connecting between these regions are necessary.

It was shown in \cite{205,206,207} that this can be done by a suitable choice of renormalization scheme. They adopted \( \overline{MS} \) scheme for everything when \( \mu_R > m_h \), but they used zero-momentum subtraction for loops with heavy quarks when \( \mu_R < m_h \), and \( \overline{MS} \) for everything else. This method gives automatic decoupling of heavy quarks, and allows calculations at scales of order
with all its mass effects taken into account by the resummation of the effects. Following them, the number of the active quark flavors in the beta function is changed by exactly one at the break point \( \mu_R = m_h \), and the coupling is made continuous there. It can be shown by explicit calculation that, at the one-loop approximation, this break point is at \( \mu_R/m_h = 1 \) and not at some other ratio, provided that \( \overline{MS} \) renormalization is used. If desired, we can calculate higher order corrections to this matching condition. (Actually, currently the condition up to \( N^3LO \) is known \([208, 209]\).) It is however not yet known how to make an accurate direct experimental measurement of a running quark mass, so we simply adjust \( m_h \) to fit a physical quantity such as the production cross section. (Therefore one should not be surprised when these masses do not exactly agree with the naive expectation of one-half of the energy of the threshold for open heavy quark production.)

In my calculation, the charm quark mass \( m_c \) is taken at \( m_c = 1.4 \) GeV, the bottom quark \( m_b = 4.5 \) GeV, and the top quark \( m_t = 174.0 \) GeV following the conventional choice of perturbative QCD analysis market.

After all, I adopted the running coupling \( \alpha_s(\mu_R^2) \) which obeys NLO evolution equation of eq. \([227]\) with \( n_f \) changing its value depending on \( \mu_F \) as

\[
n_f = \begin{cases} 
    3 : & \mu_{0,R} \sim 1\text{GeV} < \mu_R < m_c \\
    4 : & m_c < \mu_R < m_b \\
    5 : & m_b < \mu_R < m_t \\
    6 : & m_t < \mu_R 
\end{cases}, \tag{4.93}
\]

where \( m_{c,b,t} = 1.4, 4.5, 174 \) GeV. At each threshold scale, the \( \alpha_s \) is continuously connected, fig. \([4.11]\). Then the choice of \( \alpha_s(M_Z^2) = 0.1176 \) corresponds to \( \Lambda_{QCD,3} \sim 200 \) MeV, where \( \Lambda_{QCD,3} \) is the \( \Lambda \) defined in lowest \( n_f \) region which defines a typical scale where the perturbative treatment breaks down.

The complications coming from the heavy mass quarks also exist in the evolution of parton distributions through the \( n_f \) in the splitting functions. For unpolarized PDFs, the matching conditions are systematically studied \([210, 211, 167]\). Following these studies, in my NLO analysis, I continuously connected the unpolarized heavy quark distributions from zero at \( \mu_F = m_h \) where \( m_h \) is taken at the same value with that of the strong running coupling constant. The continuity also holds for the light distributions. Above the threshold, the distributions evolve with the \( n_f + 1 \) active flavors.

For fragmentation functions and helicity distributions, there seems to be no clear systematic derivation of those conditions. (Actually there are several
studies on the heavy quark production cross sections in SIA which requires the resummation of the heavy quark mass, e.g., \cite{212}. Thus in my analysis, I applied simple connecting rules for those distributions.

For fragmentation functions, I employed the scheme known as the zero-mass variable flavor number scheme \cite{211}. In that method, the massive partons are treated as being infinitely massive below the threshold $m_h$, and totally massless above the threshold, i.e., the heavy quarks turned on at the scale $\mu_F = m_h$ and $n_f$ changes to $n_f + 1$ at the threshold. As shown in the previous section, I applied discontinuous initial distributions at the threshold. I expect that the choice of the effective threshold $m_h$ would compensate potential ambiguities of the resummation effect around the scale of $\mu_F \sim m_{0,h}$ where $m_h$ is the true (pole) mass of heavy quarks. For the appearance of these distributions in the calculation of SIA cross sections, however, I took the physical pair creation condition $Q^2 > 4m_h^2$ following \cite{66}. For unpolarized SIDIS process which is related to the hadron multiplicities, the entrance occurs virtually from $\mu_F^2 > m_h^2$ similarly as unpolarized PDFs and DIS \cite{211} \cite{167}.

On the other hand, for the helicity distributions, I always kept simply three active flavors of up, down and strange distributions because of the limited kinematic region of spin-dependent processes, as is listed in subsections 5.3 and 4.6.4.
4.6 Data Used in Our Fit

In this section, I’m going to list up the experimental data used in our fit. The data in the first two subsections were applied for the fit of fragmentation functions. Those in the following two subsections were for the parton helicity distribution fit.

4.6.1 Inclusive Hadron Production in $e^+e^-$ Collision

Here I’m going to list the experimental data used in the fit of fragmentation functions of pion, kaon, and proton. Note that the following data for the charged particle are the sum of the yields of particle and anti-particle.

In the following list, there are data with heavy quark contribution enriched. These are the data extracted only the heavy quarks, bottom and charm quarks, contributions by Monte Calro study. These are also fitted to determine the fragmentation functions of these quarks after the corresponding quantities are calculated just like eqs. 3.46 and 3.47 taking only the sum of corresponding flavor.

I set the kinematic constraint on $z$ for all the applied experimental data as $0.05 < z < 0.8$ to eliminate higher twist (mass) or resummation contributions as highly small and large $z$ [148].

Some of the following data are given based on the momentum $P_h$ of the final state hadron, instead of its energy $E_h$ (ref. eq. 3.42). In that case, I converted to those of energy by the $E_h^2 = P_h^2 + m_h^2$. (For charged hadron data, I applied the mass of pion because of its dominance in the data.)

- **$\pi^{+(-)}$ production in $e^+e^-$ collision**

  Table 4.2 is a table of the data for $\pi^{+(-)}$ production in $e^+e^-$ collision compared with pion ($\pi^+$) fragmentation functions.

- **$\pi^0$ production in $e^+e^-$ collision**

  Table 4.3 is a table of the data for $\pi^0$ production in $e^+e^-$ collision compared again with pion ($\pi^+$) fragmentation functions.
• $K^{+(-)}$ production in $e^+ + e^-$ collision

Table 4.4 is the list of the data for $K^{+(-)}$ production in $e^+ + e^-$ collision compared with kaon ($K^+$) fragmentation functions.

• proton (anti-proton) production in $e^+ + e^-$ collision

Table 4.5 is the list of the data for proton (anti-proton) production in $e^+ + e^-$ collision compared with proton ($p^+$) fragmentation functions.

• charged hadron production in $e^+ + e^-$ collision

Table 4.6 is the list of the data for charged hadron $h^{+(-)}$ production in $e^+ + e^-$ collision compared with $\pi^+$, $K^+$ and $p^+$ fragmentation functions.

Table 4.2: experimental data applied in my fit: $e^+ + e^- \rightarrow \pi^{+(-)} + X$

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<th>$\sqrt{s}$ [GeV]</th>
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<td>TPC [214]</td>
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<td>SLD(c enriched) [215]</td>
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<td>ALEPH [216]</td>
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<td>OPAL [217]</td>
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<tr>
<td>DELPHI(b enriched) [218]</td>
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Table 4.3: experimental data applied in my fit: $e^+ + e^- \rightarrow \pi^0 + X$

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Table 4.4: experimental data applied in my fit: $e^+ + e^- \rightarrow K^{+(-)} + X$

<table>
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<td>29</td>
</tr>
<tr>
<td>OPAL</td>
<td>91.2</td>
<td>33</td>
</tr>
<tr>
<td>DELPHI(inclusive)</td>
<td>91.2</td>
<td>14</td>
</tr>
<tr>
<td>DELPHI(inclusive)</td>
<td>91.2</td>
<td>23</td>
</tr>
<tr>
<td>DELPHI(b enriched)</td>
<td>91.2</td>
<td>23</td>
</tr>
</tbody>
</table>
Table 4.5: experimental data applied in my fit: $e^+ + e^- \rightarrow p(\bar{p}) + X$

<table>
<thead>
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<th>$\sqrt{s}$ [GeV]</th>
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<td>4</td>
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<tr>
<td>TPC [214]</td>
<td>29</td>
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</tr>
<tr>
<td>SLD(inclusive) [215]</td>
<td>91.2</td>
<td>33</td>
</tr>
<tr>
<td>SLD(c enriched) [215]</td>
<td>91.2</td>
<td>20</td>
</tr>
<tr>
<td>SLD(b enriched) [215]</td>
<td>91.2</td>
<td>20</td>
</tr>
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<td>ALEPH [216]</td>
<td>91.2</td>
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<tr>
<td>OPAL [217]</td>
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</tr>
<tr>
<td>DELPHI(inclusive) [218]</td>
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</tr>
<tr>
<td>DELPHI(b enriched) [218]</td>
<td>91.2</td>
<td>23</td>
</tr>
</tbody>
</table>

Table 4.6: experimental data applied in my fit: $e^+ + e^- \rightarrow h^{+(-)} + X$

<table>
<thead>
<tr>
<th>Data set</th>
<th>$\sqrt{s}$ [GeV]</th>
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</thead>
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<tr>
<td>TPC [214]</td>
<td>29</td>
<td>34</td>
</tr>
<tr>
<td>SLD [215]</td>
<td>91.2</td>
<td>39</td>
</tr>
<tr>
<td>ALEPH [226]</td>
<td>91.2</td>
<td>46</td>
</tr>
<tr>
<td>DELPHI [227]</td>
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<td>OPAL(transverse) [228]</td>
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</tr>
<tr>
<td>OPAL(longitudinal) [228]</td>
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</tr>
<tr>
<td>OPAL(inclusive) [229]</td>
<td>91.2</td>
<td>22</td>
</tr>
<tr>
<td>OPAL(c enriched) [229]</td>
<td>91.2</td>
<td>22</td>
</tr>
<tr>
<td>OPAL(b enriched) [229]</td>
<td>91.2</td>
<td>22</td>
</tr>
</tbody>
</table>
4.6.2 Hadron Multiplicity in Deep Inelastic Scattering

Table 4.7 is a list of data for hadron multiplicity $\eta^H$ defined in eq. 3.65 in unpolarized semi-inclusive deep inelastic scattering. $p, d$ in the braket attached to the name of the experiment indicate the sort of the targets, and $\pi^\pm$ and $K^\pm$ denote the detected final state hadron.

$\eta^H$ is a function of $x$, $z$ and $Q^2$. Experimentally $z$ is independent on the choice of $x$ and $Q^2$ in principle. I applied $z$ dependent results in my fit calculated at the average $Q^2 = 2.5$ GeV$^2$. The corresponding $x$ is taken from [230] and set at the bin range $0.14 > x > 0.10$. I took the average of $\eta^H$ over the bin. I also set the constrain on $z$ as $0.8 > z > 0.05$ as the case of $e^+ e^-$ data.

Table 4.7: experimental data applied in my fit: hadron multiplicities $\eta^H$ defined in eq. 3.65

<table>
<thead>
<tr>
<th>Data set</th>
<th>$\sqrt{s}$ [GeV]</th>
<th>No. of data</th>
</tr>
</thead>
<tbody>
<tr>
<td>HERMES($p, \pi^+$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
<tr>
<td>HERMES($p, \pi^-$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
<tr>
<td>HERMES($p, K^+$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
<tr>
<td>HERMES($p, K^-$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
<tr>
<td>HERMES($d, \pi^+$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
<tr>
<td>HERMES($d, \pi^-$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
<tr>
<td>HERMES($d, K^+$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
<tr>
<td>HERMES($d, K^-$)</td>
<td>$\sqrt{2.5}$</td>
<td>13</td>
</tr>
</tbody>
</table>
4.6.3 Polarized Deep Inelastic Scattering

Table 4.8 is the list of data of asymmetry $A_1$ of longitudinally polarized deep inelastic scattering process with several targets used in our fit for the parton helicity distributions. $p$, $n$ and $d$ in the bracket attached to the name of experiment indicate the sort of the targets.

I set the constraints on $Q^2$ and $W^2$ as $Q^2 > 1 \text{ GeV}^2$ and $W^2 > (2.5)^2 \text{ GeV}^2$ to extract deep inelastic scattering region without $\Delta$ resonances. Fig. 4.12 shows the typical kinematic range covered by $A_1$. There I showed the proton target data only for simplicity.

![Figure 4.12: typical $x$-$Q^2$ range covered by the $A_1$ data with proton target listed in table 4.8](image)

Figure 4.12: typical $x$-$Q^2$ range covered by the $A_1$ data with proton target listed in table 4.8
Table 4.8: experimental data applied in my fit: the asymmetry $A_1$ in the longitudinally polarized DIS

<table>
<thead>
<tr>
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</tr>
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<tr>
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<td>[48]</td>
</tr>
<tr>
<td>SMC($p$)</td>
<td>[231]</td>
</tr>
<tr>
<td>E143($p$)</td>
<td>[232]</td>
</tr>
<tr>
<td>E155($p$)</td>
<td>[233]</td>
</tr>
<tr>
<td>HERMES($p$)</td>
<td>[234]</td>
</tr>
<tr>
<td>E142($n$)</td>
<td>[235]</td>
</tr>
<tr>
<td>E143($n$)</td>
<td>[232]</td>
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<tr>
<td>E154($n$)</td>
<td>[236]</td>
</tr>
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<td>[233]</td>
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<tr>
<td>HERMES($n$)</td>
<td>[237]</td>
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<tr>
<td>JLAB-E-99-117($n$)</td>
<td>[238]</td>
</tr>
<tr>
<td>SMC($d$)</td>
<td>[231]</td>
</tr>
<tr>
<td>E143($d$)</td>
<td>[232]</td>
</tr>
<tr>
<td>E155($d$)</td>
<td>[233]</td>
</tr>
<tr>
<td>HERMES($d$)</td>
<td>[234]</td>
</tr>
<tr>
<td>COMPASS($d$)</td>
<td>[240]</td>
</tr>
</tbody>
</table>

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4.6.4 Polarized Semi-Inclusive Deep Inelastic Scattering

Table 4.9 is the list of data of asymmetry $A_H^1$ of longitudinally polarized semi-inclusive deep inelastic scattering process with several targets used in our fit for the parton helicity distributions. $p$, $n$ and $d$ in the bracket attached to the name of experiment indicate the sort of the targets, and $h^\pm$, $\pi^\pm$ and $K^\pm$ denote the detected final state hadron.

The kinematic conditions for $Q^2$ and $W^2$ are the same as $A_1$ case. All the experimental data are given as an integrated yield over the detector coverage on $z$. Following the papers [241, 242], I took the partial integration of $A_H^1(x, z)$ on $z$ over $0.2 < z < 0.8$, i.e.,

$$A_H^1(x, Q^2)_{\text{measure}} = \int_{0.2}^{0.8} dz A_H^1(x, z, Q^2). \quad (4.94)$$

Fig. 4.13 shows the kinematic range covered by $A_H^1$ on $x - Q^2$ plane.

Table 4.9: experimental data applied in my fit: the asymmetry $A_H^1$ in the longitudinally polarized semi-inclusive DIS

<table>
<thead>
<tr>
<th>Data set</th>
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</tr>
</thead>
<tbody>
<tr>
<td>SMC($p, h^+$) [241]</td>
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<tr>
<td>HERMES($p, h^+$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>SMC($p, h^-$) [241]</td>
<td>12</td>
</tr>
<tr>
<td>HERMES($p, h^-$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>HERMES($p, \pi^+$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>HERMES($p, \pi^-$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>SMC($d, h^+$) [241]</td>
<td>12</td>
</tr>
<tr>
<td>HERMES($d, h^+$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>SMC($d, h^-$) [241]</td>
<td>12</td>
</tr>
<tr>
<td>HERMES($d, h^-$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>HERMES($d, \pi^+$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>HERMES($d, \pi^-$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>HERMES($d, K^+$) [242]</td>
<td>9</td>
</tr>
<tr>
<td>HERMES($d, K^-$) [242]</td>
<td>9</td>
</tr>
</tbody>
</table>
Figure 4.13: $x$-$Q^2$ range covered by the SIDIS data in table 4.9: SMC data have the same $x$-$Q^2$ for the four different type asymmetries, and $Q^2$ value of the data is averaged to 10.0 GeV$^2$ because of the negligible $Q^2$ dependence as indicated in [241].
4.7 Systematic Errors in My Fit Procedure

In this section, I’m going to list up the systematic errors which would be potentially included in my analysis. These ambiguities are investigated in my analysis after I obtained the optimized distributions and are included as the theoretical uncertainty in the final results by taking the square sum of these.

- renormalization and factorization scale dependences
  In perturbative QCD theoretical framework, there are two scales which we can choose arbitrary, i.e., the renormalization scale $\mu_R$ and the factorization scale $\mu_F$. Those scales are introduced artificially to define ultraviolet cutoff and infrared (collinear) cutoff. To avoid additional large logarithms, those scales usually set at some high energy scale $Q$ of a process of our interest, i.e., $\mu_R = \mu_F = Q$. Nonetheless any observable quantity calculated in this framework should not be dependent on the choice of those artificial scales as it was discussed in chapter 2. The dependence however appears when the perturbation is limited to finite order, like our case of the next leading order (NLO). If we proceed to higher order prediction, the dependence is expected to diminish. Thus the existing dependence can be regarded as a sort of theoretical ambiguity which the NLO prediction potentially contains.

  To estimate those ambiguities, the method usually taken is varying those scale $\mu_{R,F}$ between $2Q$ and $Q/2$ \[^{243}\]. (Actually there is no strict basis to choose the factor 2. The factor of 2 is commonly regarded as a standard variation.) In my analysis, I kept $\mu_R = \mu_F$ for simplicity. The ambiguity was investigated in my analysis in the following way.

  First I find optimized parameters with the condition of $\mu_R = \mu_F = Q$. Then set $\mu_R = \mu_F = 2Q$ or $\mu_R = \mu_F = Q/2$ and perform the fit under this condition, and examine the deviation of the determined parameters from the original ones. I consider those deviations as the theoretical ambiguity which the determined parameters or equivalently distributions contain.

- $\alpha_s$ dependence
  As I have already mentioned in section 4.5 I adopted $\alpha_s = 0.1176$ at the Z resonance scale $M_Z = 91.1876$. This value was taken from the world average \[^{13}\]. More correctly, $\alpha_s = 0.1176 \pm 0.002$ and $M_Z = 91.1876 \pm 0.0021$. Because of the logarithmic evolution of the running coupling, the error of the Z boson mass can be safely neglected. I
investigated the effect of the $\alpha_s$ by changing its value by $2\sigma = 0.004$ which roughly corresponds to the amplitude of the statistical error usually contained in perturbative QCD analyses \cite{13}. (Of course the $\alpha_s$ value should be included in my fit as a fitted parameter in more correct manner. In my analysis, I took instead this procedure for simplicity.)

- **unpolarized PDF dependence**
  As is mentioned in section 4.5 I adopted MRST 2001 \cite{60} for unpolarized PDFs because of its ease of application to my Mellin framework by virtue of those functional forms at initial scale. In addition to MRST 1998 \cite{61}, I have also implemented unpolarized PDFs given CTEQ group \cite{42} (CTEQ6). I investigated the effect of the choice of unpolarized PDF set by switching those PDF sets.

- **the effect of the errors of fragmentation function on the determination of parton helicity distributions**
  By the fit of fragmentation functions, we can estimate the (statistical and theoretical) errors of fragmentation functions. The optimized fragmentation functions are implemented in my fit for the extraction of parton helicity distributions. The effect of the errors of fragmentation functions on the determination of parton helicity distributions was investigated mainly by Lagrange multiplier method as I reported in \cite{57} with the primitive $x$ space framework. In that study I could not perform enough detailed studies in the next leading order accuracy mainly because of the barrier of the considerable increment of the computation time. In my newly constructed framework, I can perform it within compatible time scale.

- **functional form dependence**
  I always assumed the simple functional form of eq. 4.59. However, in principle, we can choose any functional form for the distributions at the initial scale. The ambiguity related to the choice would be difficult to estimate. In my analysis I tried to obtain somewhat quantitative estimation by changing the initial scale which corresponds to the change of functional form at the original initial scale.

- **the ambiguity related to the inverse Mellin transform**
  To achieve permissible computation time, I sacrificed the accuracy of integration of the inverse Mellin transform to some extent by finding reasonable values for the integration parameters introduced in section 4.1. Indeed I paid much attention on the accuracy of calculation in the fit routine, I investigated the effect of the choice of those parameters
on the final result by changing the parameters to yield more accurate calculation after I obtained the optimized parameters. In the fit of fragmentation functions, I can apply another integration method based on the steepest descent explained in section 4.1. The final results were also examined by using the integration method.

- **effect of the errors of helicity distribution constraints**
  As shown in subsection 4.1.2 I set the constraints on appropriate combinations of the first moments of the helicity distributions. I applied the values of 1.2573 and 0.575 for $SU(2)$ and $SU(3)$ symmetry origin constraints respectively. The experimental results were $1.2573 \pm 0.0028$ and $0.575 \pm 0.016$. I changed the values within their errors, and investigated its effect on the determination of the helicity distributions.
Chapter 5

Analysis Results

In this chapter, I’m going to briefly review current temporary results of my analyses.

5.1 Consistency Check of My Framework

First, I checked the consistency of my numerical calculations of DGLAP equation in several ways. I checked implemented splitting functions by checking several conservation sum rules discussed in the previous chapter and its asymptotic behaviors \[120\]. I also compared them with my \(x\) space integral results though those have limitation in the accuracy, and with those of steepest descent for fragmentation function and unpolarized PDFs. Finally we compared with the results of existing PDF package. Several groups provide grid information to allow us to recreate their PDFs over some kinematic range \(x, Q^2\) \[202\] \[65\]. By setting the conditions of DGLAP equation exactly equal to those of them, we can check the adequacy of my numerical calculations through the grid data, e.g. fig. 5.1. I observed well reliability, and could confirm the consistency within the accuracy of roughly \(10^{-2}\) or less in \(x > 10^{-2}\) where the physics of my interest exists. Note the (rather small) deviation appearing in fig. 5.1 comes from that of existing already in the initial scale (1 GeV\(^2\)). It seemed that their grid information was inconsistent as far as checking their original paper. (Otherwise, simply because of my lack of knowledge.) Whatever is the origin of the small deviation, it would be no harm in my analysis to use my results of unpolarized PDF calculation. I also checked that my results is not affected so much by the choice of unpolarized PDFs. Its quantitative investigation was included in the systematic error study. Note also that the consistency I obtained in the calculations is strongly dependent on the choice of integration path. Moreover, in actual
fitting process, we had to adjust them properly to achieve reasonable computation time without losing the accuracy so much. I also included the effect to the final results as one of the systematic errors.

Figure 5.1: Comparison of my numerical calculations of unpolarized PDFs with those of MRST1998 [61] at NLO: dashed curves are from the grid information of MRST1998 and solid curves are my calculation.
Figure 5.2: Comparison of my numerical calculations of polarized (helicity) PDFs with those of DSSV08 [56] at NLO: dashed curves are from the grid information of DSSV08 and solid curves are my calculation.

Figure 5.3: Comparison of my numerical calculations of unpolarized fragmentation functions with those of Kretzer00 [66] at NLO: dashed curves are from the grid information of Kretzer00 and solid curves are my calculation.
5.2 Results of Fit of Fragmentation Functions

In the following I’m going to briefly show the current results on the fit of fragmentation functions. I will show first the pion fragmentation functions, and second the kaon fragmentation functions.

5.2.1 Pion Fragmentation Functions

I performed the \( \chi^2 \) fit to the data of pion production in high energy \( e^+ e^- \) collision and pion multiplicities in deep inelastic scattering listed in subsections 4.6.1 and 4.6.2. For unpolarized parton distributions needed for the calculation of the hadron multiplicity, I employed MRST 2001\(^60\).

By the fit of pion fragmentation function, I obtained the parameters listed in table 5.1 with positive definite error matrix. The total number of parameters were 31 in total (18 for fragmentation function parameters and 13 for the renormalization scale parameters). The number of data were 193. The reduce \( \chi^2 \) was 1.12 against 162 degrees of freedom.

Table 5.1 is the parameters with those statistical errors given in the error matrix. Figs. 5.4 and 5.5 are fragmentation functions given in my fit. We can observe the suppression of dis-favored fragmentation functions in their second moments (although the strange behavior of bottom fragmentation currently exists).

These are calculated at 10 GeV\(^2\) and compared with Kretzer00 \(^66\) and DSS07 \(^68\). The error bands coming from the errors of the parameters were also plotted.

Fig. 5.6 shows my fit results with some of the experimental data of the \( e^+ e^- \) collision process. Fig. 5.7 exhibits the typical distribution of experimental data around the fit curve. I took the difference between the experimental data and theoretical curve, and normalized by the experimental error. The gray band means the statistical error coming from the errors of fragmentation functions. The errors of the parameters were evolved to the quantity by the error propagation eq. 4.45. Fig. 5.8 shows my fit results compared with the data of \( \pi^+ \) multiplicity on proton target. Fig. 5.9 displays the distributions of the data around the fit curve. The gray band shows the statistical error originating in the errors of fragmentation functions.

Fig. 5.10 shows the correlations between parameters. The projections of the multidimensional \( \chi^2 \) surface on to the corresponding parameter plane
Table 5.1: Table of fit parameters of $\pi^+$ fragmentation functions

<table>
<thead>
<tr>
<th>name</th>
<th>$\eta = D(2)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>$0.32 \pm 0.02$</td>
<td>$5. \pm 1.$</td>
<td>$10. \pm 2.$</td>
</tr>
<tr>
<td>$u$</td>
<td>$0.32 \pm 0.02$</td>
<td>$-1.37 \pm 0.06$</td>
<td>$0.82 \pm 0.08$</td>
</tr>
<tr>
<td>$d$</td>
<td>$0.27 \pm 0.07$</td>
<td>$-1.7 \pm 0.1$</td>
<td>$1.1 \pm 0.2$</td>
</tr>
<tr>
<td>$s$</td>
<td>$0.05 \pm 0.01$</td>
<td>$9. \pm 4.$</td>
<td>$11. \pm 5.$</td>
</tr>
<tr>
<td>$c$</td>
<td>$0.10 \pm 0.03$</td>
<td>$-1.0 \pm 0.3$</td>
<td>$1.7 \pm 0.5$</td>
</tr>
<tr>
<td>$b$</td>
<td>$0.20 \pm 0.02$</td>
<td>$-1.1 \pm 0.2$</td>
<td>$8. \pm 1.$</td>
</tr>
</tbody>
</table>

were shown. The plot was made by two dimensional Lagrange multiplier method. The axes are normalized by the error of each parameter. The inner circle corresponds to $\Delta \chi^2 = 1/4$ and $N$th outer circles correspond to $\Delta \chi^2 = N^2/4$. We can observe that the curve of $\Delta \chi^2 = 1$ surely attaches to both axes as it expected (fig. 4.8). We can directly obtain these contour curves from the error matrix given from the fit. This serves as the cross check of the positive definite error matrix. Now we can conclude that the $\chi^2$ surface around the local minimum was surely parabolic in this analysis. The important issue of looking at the contour plot is that the correlation between parameters becomes clearly visible. For example, we can see that $\alpha$ and $\beta$ parameters of gluon fragmentation function have strong (positive) correlation. Toward that direction of eigen vector, $\chi^2$ becomes flat. This contains the potential danger of making the error matrix non-positive definite. Thus the understanding of the parameter correlations has another significance for finding better functional forms other than for the application of the error matrix to error calculations.
Figure 5.4: $u$ quark fragmentation function to $\pi^+$ given at 10 GeV$^2$ compared with that of Kretzer00 [66] and DSS07 [68].

Figure 5.5: gluon fragmentation function to $\pi^+$ given at 10 GeV$^2$ compared with that of Kretzer00 [66] and DSS07 [68].
Figure 5.6: Comparison of fit results with some of the experimental data of pion production in high energy $e^+ e^-$ collision.
Figure 5.7: Distribution of experimental data of the $e^+ e^-$ collision around the fit curve, normalized by the experimental error: gray band shows the error coming from the errors of fragmentation functions.
Figure 5.8: Comparison of fit results with the data of $\pi^+$ multiplicity in deep inelastic lepton scattering on proton target. ($Q^2 = 2.5$ GeV$^2$)
Figure 5.9: Distribution of experimental data of the $\pi^+$ multiplicity around the fit curve, normalized by the experimental error: gray band shows the error coming from the errors of fragmentation functions.

Figure 5.10: Contours showing the correlations between parameters: gluon fragmentation sector and favored fragmentation sector are shown.
5.2.2 Kaon Fragmentation Functions

I performed the $\chi^2$ fit to the data of kaon production in high energy $e^+e^-$ collision and kaon multiplicities in deep inelastic scattering listed in subsections 4.6.1 and 4.6.2. For basic unpolarized parton distributions needed for the calculation of the hadron multiplicity, I again employed MRST 2001 [60].

By the fit of kaon fragmentation function, I obtained the parameters listed in Table 5.1 with positive definite error matrix. Note that I could not obtain the appropriate error matrix in the condition of moving $\alpha$ and $\beta$ of gluon fragmentation function together with others. Thus I fixed them at the values given at the local minimum. The total number of parameters were 29 in total (16 for fragmentation function parameters and 13 for the renormalization scale parameters). The number of data were 164. The reduce $\chi^2$ was 0.86 against 135 degrees of freedom.

Table 5.1 is the parameters with those statistical errors given in the error matrix. I surely observed the strange quark dominance for kaon fragmentation. Fig. 5.11 is fragmentation functions given in my fit. These are calculated at 10 GeV$^2$ and compared with Kretzer00 [66] and DSS07 [68]. I also plotted the error coming from the errors of the parameters were also plotted.

I will omit to show the plots like pion case this time. I will only mention one remark for the reason why the gluon distributions were not fixed well. The main reason seems to be the experimental errors of the $e^+e^-$ collision. The difference of the differential cross section at different energy scales is compensated by the largeness of the error, Fig. 5.12. It was already observed in an analysis only with $e^+e^-$ data. As a prospect for this analysis, I expected that it was resolved by including the multiplicity data accumulated at greatly lower energy scale in sense of $e^+e^-$ collision data. It was remarkable that I could not fix them even in this analysis. Further investigations are needed on this issue.
Table 5.2: Table of fit parameters of $K^+$ fragmentation functions

<table>
<thead>
<tr>
<th>name</th>
<th>$\eta = D(2)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>0.02 ± 0.02</td>
<td>11.05 (fixed)</td>
<td>6.66 (fixed)</td>
</tr>
<tr>
<td>$\bar{s}$</td>
<td>0.31 ± 0.09</td>
<td>-0.3 ± 0.4</td>
<td>2.0 ± 0.4</td>
</tr>
<tr>
<td>$u$</td>
<td>0.065 ± 0.006</td>
<td>-1.1 ± 0.2</td>
<td>0.8 ± 0.2</td>
</tr>
<tr>
<td>$d$</td>
<td>0.020 ± 0.008</td>
<td>-1.0 ± 0.5</td>
<td>5.0 ± 2.0</td>
</tr>
<tr>
<td>$c$</td>
<td>0.10 ± 0.02</td>
<td>-0.2 ± 0.3</td>
<td>3.5 ± 0.5</td>
</tr>
<tr>
<td>$b$</td>
<td>0.0592 ± 0.006</td>
<td>-0.2 ± 0.3</td>
<td>8.0 ± 1.0</td>
</tr>
</tbody>
</table>

Figure 5.11: $u$ quark fragmentation function to $K^+$ given at 10 GeV$^2$ compared with that of Kretzer00 [66] and DSS07 [68]
Figure 5.12: Comparison of fit results with some of the experimental data of kaon production in high energy $e^+ e^-$ collision.
5.3 Results of Fit of Parton Helicity Distributions

In this subsection, I will show the current results of the fit of parton helicity distributions. For this fit, I included the asymmetry data of polarized deep inelastic lepton scattering (pDIS) and polarized semi-inclusive deep inelastic scattering (pSIDIS), which were listed in subsections 4.6.4 and . For the fragmentation functions needed for the calculations of SIDIS quantities, I applied those determined in the previous section. (The data of the asymmetry of charged hadron were not included in this analysis.)

By the fit of parton helicity distributions, I obtained the parameters listed in table 5.1 with positive definite error matrix. Note that I could not obtain the appropriate error matrix in the condition of moving $\alpha$ and $\beta$ of gluon fragmentation function together with others. Thus I fixed them at some artificial values. Further studies are needed to resolve this issue. (The only choice seems to include other gluon sensitive processes.) In this short study, I only moved its amplitude.

The total number of parameters in this fit was 17 including $\epsilon_{SU(3)}$. The number of data were 300. The reduce $\chi^2$ was 0.95 against 283 degrees of freedom.

Table 5.3 is the parameters with those statistical errors given from the error matrix. For the first moments of up and down quark, the errors are calculated with the error matrix through the constraints eqs. 4.91,4.92. We can also calculate $\Delta \Sigma$ through these results. I obtained $\Delta \Sigma = 0.16 \pm 0.05$ at the initial scale (1 GeV$^2$). On the other hand, gluon contribution $\Delta g(1)$ still contains large error even though both $\alpha$ and beta were fixed.

Figs. 5.4 and 5.5 are the helicity distributions given in my fit with statistical error bands. The helicity distributions given by DNS group (DNS05) [64] were also plotted. (They also concentrated on the analysis with DIS and SIDIS data.) I compared them at the initial scale.

Fig. 5.16 shows my fit results with some of the experimental data of $g_1$ data (not $A_1$) to see the scaling violation clearly. Fig. 5.17 exhibits the typical distribution of experimental data around the fit curve. I took the difference between the experimental data and theoretical curve, and normalized by the experimental error. The gray band means again the statistical error.
Table 5.3: Table of fit parameters of parton helicity distributions

<table>
<thead>
<tr>
<th>name</th>
<th>$\eta = \Delta q(1)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>$0.87 \pm 0.05$</td>
<td>$0.258 \pm 0.053$</td>
<td>$3.52 \pm 0.20$</td>
</tr>
<tr>
<td>$d$</td>
<td>$-0.43 \pm 0.15$</td>
<td>$0.29 \pm 0.12$</td>
<td>$5.4 \pm 1.0$</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>$-0.088 \pm 0.058$</td>
<td>$2.4 \pm 1.2$</td>
<td>$16.4 \pm 9.9$</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>$-0.048 \pm 0.028$</td>
<td>$4.3 \pm 3.5$</td>
<td>$26. \pm 27.$</td>
</tr>
<tr>
<td>$\bar{s}$</td>
<td>$-0.073 \pm 0.0080$</td>
<td>$10.2 \pm 4.2$</td>
<td>$32. \pm 15.$</td>
</tr>
<tr>
<td>$g$</td>
<td>$0.13 \pm 0.35$</td>
<td>$2.0 \text{ (fixed)}$</td>
<td>$6.0 \text{ (fixed)}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>name</th>
<th>$\delta_u$</th>
<th>$\delta_d$</th>
<th>$\epsilon_{SU(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_u$</td>
<td>$1.13 \pm 0.14$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta_d$</td>
<td>$3.07 \pm 0.81$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\epsilon_{SU(3)}$</td>
<td>$0.15 \pm 0.14$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

coming from the errors of the helicity distributions. Fig. 5.18 compares my fit results with applied $A_1^{F+}$ (proton target) data. The error band was also shown.

Figure 5.13: $u$ quark parton helicity distribution given at 1 GeV$^2$ with the error band. It is compared with that of DNS05 [64].
Figure 5.14: $d$ quark parton helicity distribution given at $1 \text{ GeV}^2$ with the error band. It is compared with that of DNS05 [64].

Figure 5.15: $s$ quark parton helicity distribution given at $1 \text{ GeV}^2$ with the error band. It is compared with that of DNS05 [64].
Figure 5.16: Comparison of fit results with some of the experimental data of $g_1$. 

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Figure 5.17: Distribution of experimental data of $g_1$ around the curve given from fitted parameters, normalized by the experimental error: gray band shows the error coming from the errors of fragmentation functions.
Figure 5.18: Comparison of fit results with the $\pi^+$ asymmetry $A_1^{\pi^+}$ data (proton target); the red band is the error from the errors of fitted parameters.

5.3.1 Effect of The Error of Fragmentation Functions to SIDIS Calculation

The calculations of semi-inclusive deep inelastic scattering quantities require the information on the fragmentation functions. As shown in the analyses in section 5.2, the determined fragmentation functions contain statistical and theoretical (systematic) errors. This time I concentrated only on the statistical errors given in section 5.2.

If the fragmentation functions changes within the error, the optimized parton helicity distributions also changes through the fit with the semi-inclusive data by finding new local minimum. Before going to quantitative estimation of the effect, it would be good to investigate roughly its impact on the determination of the helicity distributions.

If these two functions are really independent, the change of fragmentation functions does not affect the determination of the helicity distributions. It can be reinterpreted as the $\chi^2$ value needed for the determination of the helicity distributions keep unchanged even if the helicity distributions are fixed. It would be intuitively expected that the change of the $\chi^2$ value be-
comes larger when the correlation is strong.

Thus I like to see the response of the $\chi^2$ which connecting helicity distributions to fragmentation functions. In this analysis, it is the $\chi^2$ calculated for SIDIS data, $\chi^2_{\text{SIDIS}}$. On the other hand, the amplitude of the change of fragmentation functions would be comprehensively expressed as the change of $\chi^2$ for the fit of fragmentation functions, $\chi^2_{\text{FF}}$, which is independent of the helicity distribution. Therefore, I applied Lagrange multiplier method, taking the following $\Psi^2$ (eq. 4.47) for this investigation.

$$
\chi^2_{\text{total}}(a_{\text{FF}}, \lambda) = \chi^2_{\text{FF}}(a_{\text{FF}}) + \lambda \chi^2_{\text{SIDIS}}(a_{\text{FF}}, a_{\text{pPDF}}|\text{fixed}),
$$

(5.1)

where $a_{\text{FF}}$ and $a_{\text{pPDF}}$ are parameters to define the fragmentation functions and parton helicity distributions respectively. The $\chi^2_{\text{FF}}$ is the $\chi^2$ calculated in the fit of fragmentation functions, and $\chi^2_{\text{SIDIS}}$ is the semi-inclusive part of the $\chi^2$ calculated in the fit of the helicity distributions.

By fixing the helicity distributions, i.e., making $a_{\text{pPDF}}$ fixed, I investigated the response of $\chi^2_{\text{SIDIS}}$ on the change of $\chi^2_{\text{FF}}$. Fig. 5.19 shows the result of this analysis. I investigated the effect both for pion fragmentation and kaon fragmentation. From fig. 5.19 for the change (or statistical fluctuation) of fragmentation functions within the error band, i.e., $\Delta \chi^2_{\text{FF}} = 1$, $\Delta \chi^2_{\text{SIDIS}}$ changes roughly 0.1. Therefore we could conclude that the fragmentation functions are weakly correlated to helicity distributions through the SIDIS process.

Considering the contribution of the change to the total $\chi^2_{\text{TOT}} = \chi^2_{\text{DIS}} + \chi^2_{\text{SIDIS}}$ of the helicity distribution determination, it is expected that the change of fragmentation function with in its statistical error does not affect so much the determination of the helicity distributions. Compared with the typical size of the statistical fluctuation of the helicity distributions, $\Delta \chi^2_{\text{TOT}} = 1$, the change is expected to be limited well within the statistical errors of helicity distributions although further study for more quantitative estimation would be needed.
Figure 5.19: The result of Lagrange multiplier method for the investigation of the impact of fragmentation functions on helicity distribution determination.
Chapter 6

Conclusion

Quantum Chromodynamics (QCD) is a part of the standard model of the elementary particles. QCD is an unbroken non-Abelian gauge theory of the strong interaction based on local gauge symmetry of $SU(3)$. The freedom of inner $SU(3)$ is called color. Fundamental ingredients of QCD are quarks and gluons, generally called partons. A quark is a spin $1/2$ fermion carrying color triplet index, and a gluon is a spin 1 vector gauge boson carrying color octet index.

Because of its non-Abelian nature, QCD has a peculiar feature of asymptotic freedom. This enables us to calculate high energy interactions between quarks and gluons within the perturbative treatment. This is called perturbative QCD. On the other hand, in the low energy region of less than 1 GeV, QCD shows color confinement phase. This confines all the color freedoms of quarks and gluons into the spectrum of color singlet hadrons in physically observable asymptotic states. The process of the hadronization cannot be treated perturbatively.

However, in the effective description of high energy interactions including hadrons, the difficulty of the existence of the color confinement can be effectively avoided by the introduction of several parton distribution functions. Low energy behavior, or equivalently long distance behavior, of quarks and gluons is absorbed into the definition of those distributions. The extraction of the long distance behavior can be performed systematically by the elimination of infrared divergences. Most importantly the long distance part can be factored out from the rest of the parton interaction describing high energy, or short distance, parton interaction which can be treated in the perturbative procedure. The distributions contain all the long distance information on the parton structure of hadrons. This factorial separation between the short
and long distance behavior is called factorization. The factorization property is proved in many high energy processes based on the general property of infrared behavior of quarks and gluons in perturbative QCD. By virtue of the generality, the parton distributions has process independent nature. The process independence is called universality of the parton distributions.

Because of the generality of the factorization and the universality, several high energy interactions in wide range of kinematics can be analyzed in a general framework of perturbative QCD. This is called global perturbative QCD analysis. We can extract the parton distributions from various experimental data and study the inner parton structure of hadron through the distributions.

First, I tried to understand firmly the general property of perturbative QCD and its applicability to various high energy processes. Then, I constructed the new numerical calculation framework based on the perturbative QCD with the Mellin transform technique. The Mellin transformation is equivalent to the Laplace or Fourier transformation. I implemented effective Mellin-based calculation method on the foundation of the general property of the Laplace transform. With this framework, I succeeded to do several $\chi^2$ fits to experimental data of various processes within a reasonable computation time. It became about one thousand times shorter than the time needed in my previous framework where more intuitive calculation method was used in stead of the Mellin transformation. The properties of the $\chi^2$ fit were studied to evaluate statistical significance of the results appropriately. Based on the study, in our framework, statistical errors coming from experimental errors are treated properly by Hessian method or Lagrange multiplier method.

I applied the general framework to the analyses of the experimental data; 1) inclusive hadron production in high energy $e^+ e^-$ collision, 2) semi-inclusive measurement of deep inelastic lepton scattering, and 3) inclusive and semi-inclusive measurement of deep inelastic lepton scattering on longitudinally polarized target.

From the analysis of the first two processes, we can effectively extract the parton fragmentation functions. The distributions provide us the information on the number of hadrons created from a parton in the hadronization process. The hadron production in high energy $e^+ e^-$ collision (1) is a pure process where the fragmentation functions directly show up. In addition to the data, I also included data of hadron multiplicity in unpolarized deep inelastic scattering (2). The inclusion of the additional data plays significant
role to separate quark and anti-quark fragmentation functions effectively, which was impossible only with the original data.

As results, I obtained the following results for the second moments of the fragmentation functions calculated at the initial scale $\mu_F^2 = 1$ GeV$^2$ for light quarks, $\mu_F^2 = 1.4^2$ GeV$^2$ for charm quark, and $\mu_F^2 = 4.5^2$ GeV$^2$ for bottom quark; for pion fragmentation functions $D_{q}^{\pi^+}$ and $D_{g}^{\pi^+}$,

\[
\begin{align*}
D_{u,d}(2) &= 0.31 \pm 0.02 \\
D_{d,\bar{d}}(2) &= 0.27 \pm 0.07 \\
D_{s,\bar{s}}(2) &= 0.05 \pm 0.01 \\
D_{c,\bar{c}}(2) &= 0.10 \pm 0.03 \\
D_{b,\bar{b}}(2) &= 0.20 \pm 0.02 \\
D_{g}(2) &= 0.32 \pm 0.02,
\end{align*}
\]

and for kaon fragmentation functions $D_{q}^{K^+}$ and $D_{g}^{K^+}$,

\[
\begin{align*}
D_{s}(2) &= 0.31 \pm 0.09 \\
D_{u}(2) &= 0.06 \pm 0.01 \\
D_{d,d,s,\bar{s}}(2) &= 0.02 \pm 0.01 \\
D_{c,c}(2) &= 0.10 \pm 0.01 \\
D_{b,b}(2) &= 0.06 \pm 0.01 \\
D_{g}(2) &= 0.02 \pm 0.02.
\end{align*}
\]

From the analysis of the last processes, we can determine parton helicity distributions in detail with the help of the fragmentation functions. The helicity distributions yield the information on the helicity contribution of a parton to the spin of a hadron, especially a proton. Through the distribution, we can investigate the helicity contribution of each parton. This provides a clear picture of the proton spin structure. This helps up to solve the problem of the spin puzzle which was firstly indicated by EMC experiment. The result will supply a standard for further studies on the angular momentum contributions of partons through generalized parton distributions.

Inclusive measurement of deep inelastic lepton scattering on longitudinally polarized target (3) provides the shortest approach to the helicity distribution. As addition, I employed the data of semi-inclusive measurement of polarized deep inelastic scattering. The additional process ensures again the effective separation of the distributions through the information on the
As the final results, I obtained the following results at the initial scale $\mu_R^2 = 1 \text{ GeV}^2$ for the first moments of the parton helicity distributions;

\[
\begin{align*}
\Delta u &= 0.87 \\
\Delta d &= -0.43 \\
\Delta \bar{u} &= -0.09 \pm 0.06 \\
\Delta \bar{d} &= -0.05 \pm 0.03 \\
\Delta s &= \Delta \bar{s} = -0.07 \pm 0.01 \\
(\Delta \Sigma &= 0.16 \pm 0.05) \\
\Delta g &= 0.13 \pm 0.36 \\
\epsilon_{SU(3)} &= 0.15 \pm 0.14.
\end{align*}
\]

After all, the quark helicity contributions to the proton spin sums up to $16 \pm 5\%$. 

\[\text{155}\]
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I would like to thank my supervisor, Prof. Toshi-Aki Shibata, for encouraging my study. He gave me opportunity to study this field of physics, a precious chance to work abroad and gave me helpful guidance throughout my doctor curse study. With his experience in experimental and theoretical physics, he provided me detailed advices. I would also like to express my best appreciation to Assistant Prof. Yoshiyuki Miyachi. This work was based on influential discussions and computing works with him. Without his help, I could not accomplish this challenging subject. The supports of these two main collaborators to my work were remarkably huge.

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Throughout my study, the beauty of nature occasionally comes up to my mind specially when I am sinking in deep consideration. It always impresses and helps me to grow up my tiny seed of interest, along with the great works of grand scientists and admired literature writings. I finally want to thank you to this nature for its beauty.
Appendix A

Definition of Spin and Spin-Statistics Theorem

In this appendix, I would like to introduce the general definition of spins of particles in relativistic field theory. In the preceding section, I also introduce the famous spin-statistics theorem, which was first revealed in [244], based on the spin definition. The understanding of these fundamental nature of spin gave rigid ground not only for this analysis but also for catching up the idea of supersymmetry [245], which would become one of the key issues in coming LHC experiment.

A.1 Definition of Spin in Relativistic Field Theory

Invariance (symmetry) of a system, i.e. substantial invariance of the motions in the system, under a transformation derives from the invariance of the action of the system. Fundamental requirement to the system of relativistic field theories is the invariance under Poincare (continuous) transformation. The transformation can be described by the generators of its infinitesimal transformations, $P^\mu$, $J^{\mu\nu}$, standing for those of translation and Lorentz transformation respectively. These generators satisfy the following commutation relations.

$$ [P^\mu, P^\nu] = 0 $$  \hspace{1cm} (A.1)

$$ [J^{\mu\nu}, P^\rho] = i \left( g^{\rho\nu} P^\mu - g^{\rho\mu} P^\nu \right) $$  \hspace{1cm} (A.2)

$$ [J^{\mu\nu}, J^{\rho\sigma}] = i \left( g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} + g^{\mu\sigma} J^{\nu\rho} - g^{\nu\sigma} J^{\mu\rho} \right) $$  \hspace{1cm} (A.3)

where brackets represents Poisson bracket in field theory or commutator in quantum field theory (QFT).
To see what requirements are imposed on the ingredient fields in a theory under a symmetry, let $\phi_i(x)$ ingredient fields of the system. Then, the requirement of the invariance under the transformation $\phi'(x') = \hat{R} \[ \phi(x) \] \ (x' = R(x))$ imposes $S'[\phi'] = A S[\phi]$, with $A$ an arbitrary const., on the action

$$S[\phi] = \int d^4x \mathcal{L}(\phi(x), \partial\phi(x)) \left( \frac{\delta S[\phi]}{\delta \phi_i} = \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} = 0 \right). \quad (A.4)$$

this reads for the Lagrangean

$$\mathcal{L}'(\phi'(x')) \ dx^4 = A \mathcal{L}(\phi(x)) \ dx^4. \quad (A.5)$$

In case of Poincare symmetry, Jacobian is unity. Therefore this is fulfilled in case that the Lagrangean is scalar (field) under the transformation, i.e.

$$\mathcal{L}'(x') = \mathcal{L}(x). \quad (A.6)$$

Thus ingredient fields should be combined so as to make the Lagrangean invariant scalar. Because it is obvious that it is satisfied under translation transformation in the local field theory, the question reduces to the classification of the possible type of transformations of ingredient field under Lorentz transformation, i.e. irreducible representations of Lorentz transformation.

The representation becomes obvious when the generators of Lorentz transformation are decomposed into operators $A, B$ as follows.

$$J^i \equiv \frac{1}{2} \epsilon_{ijk} J^{jk} : \text{rotation,} \quad K^i \equiv J^{i0} : \text{boost} \quad (A.7)$$

$$\mathbf{J} = A (\otimes I) + (I \otimes) B \quad (A.8)$$

$$i \mathbf{K} = A (\otimes I) - (I \otimes) B \quad (A.9)$$

it follows from the definition that $A, B$ satisfy the following well-known relations of $SU(2)$ Lie algebras independently.

$$[A_i, A_j] = i \epsilon_{ijk} A_k \quad (A.10)$$

$$[B_i, B_j] = i \epsilon_{ijk} B_k \quad (A.11)$$

$$[A_i, B_j] = 0 \quad (A.12)$$

Therefore possible ingredient fields in relativistic field theory are expressed by the fundamental fields of $\psi^{(A,B)}(x)$, which are classified by two independent positive integer or half-integer $A, B$.

$$\psi^{(A,B)}_{a,b} \quad (a = -A, -A + 1, \ldots, A - 1, A \ b = -B, \ldots, B)$$

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which have \((2A + 1)(2B + 1)\) dimension and transform along with the corresponding representations \(D^{(A,B)}[J]\) under (infinitesimal) Lorentz transformation. The operation of \(J_{\mu\nu}\) on a representation \(\psi^{(A,B)}(x)\) then becomes

\[
[J_{\mu\nu}, \psi^{(A,B)}(x)] = i \left( L_{\mu\nu}\delta^i_j + (D^{(A,B)}[J_{\mu\nu}])^i_j \right) \psi^{(A,B)}(x).
\] (A.13)

where \(L_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu)\), which is regarded as the spacial part of Lorentz transformation. Hence the matrix \(D^{(A,B)}[J]\) for the rotation part of \(J_{\mu\nu}\) is naturally recognized as inner rotation, i.e. spin, of the field. Because \(J = A + B\), the possible magnitudes of spin \(s\) included in \(\psi^{(A,B)}\) vary over \(A + B, A + B - 1, \ldots, |A - B|\). For example, \((0, 0)\) represents scalar field with \(s = 0\), \((0, 1/2)\) denotes spinor with \(s = 1/2\), and \((1/2, 1/2)\) is vector with \(s = 1, 0\).

Because \(J^2\) is, however, three-vector scalar and the value \(s\) varies under Lorentz boost, it is convenient to have Lorentz invariant expression of the spin just like mass. Noting that Lorentz invariance of mass comes from the fact that \(P^2\) is a Casimir operator of the Poincare group, this becomes possible by the definition of the new Casimir operator \(W^\mu\);

\[
W^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma} P_\rho J_{\sigma}\quad\text{(Pauli-Lubanski vector)}
\] (A.14)

which satisfies the following commutation relations.

\[
[W^\mu, J^{\rho\sigma}] = i \left( g^{\mu\rho} W^\sigma - g^{\mu\sigma} W^\rho \right) : W^\mu \text{ is surely vector}
\] (A.15)

\[
[W^\mu, P^\nu] = 0 : W^\mu \text{ has same eigen states as } P^\mu
\] (A.16)

\[
[W^2, P^\nu] = [W^2, J^{\rho\sigma}] = 0 : W^2 \text{ is invariant in Poincare trans.}
\] (A.17)

\[
[W^\mu, W^\nu] = i \epsilon^{\mu\rho\sigma} W_\rho P_\sigma
\] (A.18)

These equations make \(W^\mu\) a generator of little (sub) group of Lorentz group, which is defined as that of transformations which keep the eigenvalue of \(P^\mu\) unchanged, i.e. \(U(W)^\mu_\mu p^\mu = p^\mu\). (note also \(W^\mu P_\mu = 0\). The representations of the little group can be largely classified with "standard" momentum \(k^\mu\) as \(U(W)^\mu_\mu k^\mu = k^\mu\). For the case \(P^2 = M^2 > 0\), \(k^\mu = (\pm M, 0)\) and for \(P^2 = 0\), \(k^\mu = (\pm \kappa, 0, \kappa)\) and so on. ) This property is essential for the classification of the representation of Poincare group. This aspect of the property of \(W^\mu\) is well summarized in [92].

Because the structure of the last equation (A.18) changes according to the representation of \(P^\mu\), I restrict the following discussion in the case of \(P^2 = M^2 > 0\) and \(M > 0\) for simplicity, (indeed physically meaningful representations corresponds \(P^2 \geq 0\) with \(p^0 > 0\)). To see the concrete meaning,
of (A.18) as an operator to a field, I introduce an eigenstate of the operator $P^\mu$, $|p\rangle$ ($P^\mu|p\rangle = p^\mu|p\rangle$). In case $M^2 > 0$, we can generally take its “rest frame” defined by $p^\mu = (M, 0)$. After the state of the rest frame is operated from its right on both sides, eq. (A.18) then recovers $SU(2)$ structure originated from $J$. 

$$W^\mu = (0, mJ)$$ \hspace{1cm} (A.19) \\
$$[W_i, W_j] = i m \epsilon_{ijk}W_k$$ \hspace{1cm} (A.20)

Therefore, it turns out that the invariant $W^2$ has eigen value of $-m^2s(s+1)$ for the state $|p\rangle$. Additionally we can also define a Lorentz invariant projection of $W^\mu$, which corresponds to $J_z$ in usual sense. Especially, if we introduce a polarization four-vector $n^\mu$ as

$$n^\mu = \left( \frac{|p|}{M^2}, \frac{(p_0^2 - p^2)}{2M^2|p|} \right) \hspace{1cm} (p^\mu n_\mu = 0, \quad n^2 = -M^{-2}),$$

then the projection of $W^\mu$ on $n_\mu$ becomes

$$h \equiv -n_\mu W^\mu = \frac{J \cdot p}{|p|}. \hspace{1cm} (A.21)$$

This is nothing but helicity, which has eigen values of $-s, -s+1, \ldots, s$. Thus helicity is Lorentz invariant quantity still indicating the size of the spin onto the direction of $p$.

As a conclusion, we arrive at the proper definition of spin in relativistic field theory with the operators of $W^\mu$ and $h$, and we can assign a general eigen states of a relativistic field theory as $|m^2, p, s(s+1), \sigma, \ldots\rangle$ with $\sigma$ an eigenvalue of the helicity operator and “…” other quantum numbers peculiar to the theory.

Finally let me give short notations. First, in the above general statements, we did not consider the state $|p\rangle$ as a one-particle state. Those statements and the result of classification of a state by $|m^2, p, s(s+1), \sigma, \ldots\rangle$ equally hold for one-particle state which appears as an asymptotic state of $|p\rangle$. An important fact is that the one-particle state is not necessarily of elementary nature, but can be some bound states in the theory, like hadrons in QCD. Second, in case that $P^2 = 0$, the the structure of $W^\mu$ becomes two dimensional Euclidean group $E_2$, i.e. translation to two direction and rotation on the plane. Indeed the value of $s$ can be restricted into integer or half-integer even in this case from the point of view of global simply-connected structure of the Lorentz
group, the value of the helicity is restricted to only one value $s$ or $-s$ as the results of one rotational direction in $E_2$. (For massless particle, left and right helicity states should be considered as different states even if a theory requires the existence of both states, like photon in QED.)

### A.2 Spin-Statistics Theorem

The first comprehension that an electron has spin $1/2$, dates back to early 18th century, stimulated by the famous Stern-Gerlach experiment. An electron was also known as a fermion which obeys Pauli exclusion principle. To handle the spin $1/2$ quantum number in quantum field theory, the wave function of electron extended to two component one which transform along with fundamental representation of $SU(2)$ under space rotation. This extension which seems artificial was the first perception of multi-element field also for fermions, just like electro-magnetic field which was known to have spin 1 tensor (or vector) wave function. In the non-relativistic field theory, however, these is no fundamental theoretical basis to impose some limited spin value on fermions, i.e. fermion may have any other spin other than $1/2$.

The first realization that a natural derivation of the relation between the type of field, i.e. the type of spin (as mentioned in previous section), and statistics (boson and fermion) becomes possible in relativistic quantum field theory was given by W. Pauli [244]. This is known as spin-statistics theorem. In the following two subsections, I’m going to briefly introduce the theorem. First is for showing the limitation of non-quantized relativistic theory, and second is for revealing the central idea of the theorem with the help of the well summarized discussion in [246]. I assume the particles is massive for simplicity in the second subsection. Instead I will give short remarks on the potential difficulty for some massless particles in the last subsections with related interesting subjects.

#### A.2.1 limitation of non-quantized relativistic field theory

Consider arbitrary free one-particle state $\varphi(x)$, which appears as an asymptotic free field of $\psi(x)$ of the previous section in the limit of $t \to \infty$ in a field theory. $\varphi(x)$ obeys free field equation, which should also be derived asymptotically from the original field theory. As mentioned in the previous section, the possible spin state of a field $\varphi(x)$ is uniquely pinned down by the two integer or half-integer parameter ($A, B$). In the following, two cases,
where \( s \) is integer or half-integer, are separately treated. Then it becomes clear that the inconsistency inevitably exists for keeping particle nature of the field \( \psi(x) \) in non-quantized relativistic field theory for any spin.

Before going to the cases, let me give short remark on the property of the direct product of \( \varphi(x) \), \( \varphi \otimes \varphi' \), which plays central role in the following cases. Reminding the operators \( A \) and \( B \) in the previous section implemented for the classification of \( \psi \) by \((A, B)\) are independent \( SU(2) \) operators, it would be easily clear that the fields created by the direct product of \((A, B)\) and \((A', B')\) behaves under Lorentz transformation as \((A'', B'')\), where \( A'' \) spans over \( A + A', A + A' - 1, \ldots, |A - A'| \) and \( B'' \) over \( B + B', \ldots, |B - B'| \).

**a) case that spin \( s \) is integer**

The possible combinations of \((A, B)\) is categorized into two cases, both \( A \) and \( B \) are integer or half-integer. Let \( \varphi(x) \) expressed by \( U^+, U^- \) according to the cases, named “+1 class” and “−1 class” respectively.

\[
(A, B) = (\text{int, int}) \implies \varphi \rightarrow U^+ \ (\text{scalar, etc.}) : +1 \ \text{class},
\]

\[
(A, B) = (\text{half-int, half-int}) \implies \varphi \rightarrow U^- \ (\text{vector, etc.}) : -1 \ \text{class}.
\]

We can also set \((U^\pm)^* = U^{\mp}\) because the complex conjugate operation is identified as the interchange between \( A \) and \( B \) in \((A, B)\), which would be obvious noting the definition of \( J \) and \( K \) by the operators of \( A \) and \( B \) in the previous section.

Any fields created by the direct products of two \( U^+ \)'s or \( U^- \) becomes categorized as +1 class following the previous remark, whereas the product of \( U^+ \) with \( U^- \) contains only −1 class. This is summarized in the table A.1, where \((-)\epsilon \text{ class} \) will be defined in the next case.

Then let us now consider general linear free field equations of the \( U \)'s, which does not necessarily have to be of the first order. Noting that the differentiation \( \partial / \partial x^\mu \), expressed simply \( k \), is a vector, categorized in −1 class, we can generally express the equations by each of the following two
forms along with the idea of $-1$ and $+1$ classes.

\begin{align}
\sum k U^+ &= \sum U^- \\
\sum k U^- &= \sum U^+, 
\end{align}

(A.25) 
(A.26) 
(A.27)

where I have omitted an even number of the $k$ factors. For example, the Klein-Gordon equation of a scalar field $\phi$, $\Box \phi - m^2 \phi = 0$, is expressed as $\sum U^+ = 0$, so categorized as the second case. The Maxwell equations of the vector potential $A^\mu$ and auxiliary scalar field $B$ in the covariant gauge, $\Box A^\mu = \partial^\mu (\partial_\nu A^\nu) + \partial^\mu B$ and $\partial^\mu A_\mu = -\alpha B$, are expressed as $U^- = \sum k U^+$ and $k U^- = U^+$ respectively.

We find that these equations keep invariant under the transformation of

\begin{align}
k \rightarrow -k, \quad U^+ \rightarrow U^+, \quad \text{and} \quad U^- \rightarrow -U^-,
\end{align}

(A.28)

that is to say, if a solution of these equations exits, the equations contain the quantities given from the solution by the transformation as its solution too.

Here we consider tensors $T$ of even rank (like scalar) and $S$ of odd rank (like vector), both of which are composed quadratically or bilinearly of the U’s. The general form of these tensors can be expressed as follows.

\begin{align}
T &= \sum U^+ U^+ + \sum U^- U^- + \sum U^- k U^+, \\
S &= \sum U^+ k U^+ + \sum U^- k U^- + \sum U^- U^-.
\end{align}

(A.29) 
(A.30)

$T$ and $S$ transforms under transformation (A.28) as

\begin{align}
T \rightarrow T, \quad S \rightarrow -S.
\end{align}

(A.31)

In case that $S$ is current vector $j^\mu$ of a free field, this says it is impossible to keep positive definite charge density $j^0$ with the fields of integer spin. In case of non-quantized field, the density is interpreted as the probability density to find a particle describe by the field. Thus we can conclude that it is impossible to impose a particle picture on integer spin fields.
The example of fields of those class are in the Dirac field \( \psi(x) = (\xi(x), \eta(x)) \) in the spinor representation. The upper two component field \( \xi(x) \) is an example of \(-\epsilon\) class and \(\eta(x)\) for \(+\epsilon\) class. This time, the complex conjugate operator interchanges the classes, i.e. \((U^{\pm\epsilon})^* = U^{\mp\epsilon}\). It would be easy to find the direct product properties shown in table A.1 between these classes.

The linear free field equations of the field of these classes can be expressed comprehensively as

\[
\sum k U^{+\epsilon} = \sum U^{-\epsilon} \quad \text{(A.34)} \\
\sum k U^{-\epsilon} = \sum U^{+\epsilon}. \quad \text{(A.35)}
\]

The Dirac equation, as an example, \((i \not\! \nabla - m) \psi = 0\) can be decomposed into two equations of \(i \sigma^\mu \partial_\mu \eta = m \xi\) and \(i \bar{\sigma}^\mu \partial_\mu \xi = m \eta\). Those are expressed as \(k U^{+\epsilon} = U^{-\epsilon}\) and \(k U^{-\epsilon} = U^{+\epsilon}\) respectively.

We see these equations are invariant under the transformation of

\[
k \to -k, \quad U^{+\epsilon} \to i U^{+\epsilon}, \quad \text{and} \quad U^{-\epsilon} \to -i U^{-\epsilon}, \quad \text{(A.37)}
\]

where the additional factor \(i\) comes to treat properly the interchange effect by the complex conjugate.
We consider again tensors $T$ and $S$, composed by field of these classes.

\[ T = \sum U^{+\epsilon}U^{+\epsilon} + \sum U^{-\epsilon}U^{-\epsilon} + \sum U^{-\epsilon}kU^{+\epsilon}, \quad (A.38) \]
\[ S = \sum U^{+\epsilon}kU^{+\epsilon} + \sum U^{-\epsilon}kU^{-\epsilon} + \sum U^{-\epsilon}U^{+\epsilon}. \quad (A.39) \]

Then $T$ and $S$ transforms under transformation [A.37] as

\[ T \rightarrow -T, \quad S \rightarrow S. \quad (A.40) \]

This property under the transform endangers the particle nature of a field in case $T$ is the energy-momentum tensor. A free field with half-integer spin inevitably has the solution with negative energy. Thus we again observed the indefinite particle picture even for half-integer spin fields.

After all, we saw that non-quantized relativistic field theory corrupts in describing the particle picture by a field, unlike non-relativistic field theory like Schrodinger equation. These observation was the generalization of the difficulties of the particle interpretation based on wave functions, which had already known for Dirac and Klein-Gordon field (ref. [247]). Therefore, to keep the duality nature between wave and particle picture without any inconsistency, we are unavoidably urged to the quantization of the relativistic field theory. We will see in the following subsection that the relativistic quantum field theory naturally derives the relation between spin and statistics.

### A.2.2 proof of spin-statistics theorem

As mentioned before, I’m going to treat massive particles only in this subsection. Fermions and bosons are distinguished by the statistics to which they obey, Fermi-Dirac and Bose-Einstein statistics. In quantum theory, those statistics can be expressed and discriminated each other by anti-commutation $[\alpha, \beta]_+$ or commutation $[\alpha, \beta]_-$ relations between creation and annihilation operators of corresponding particles. In general, there seems no guiding principle for the choice of the commutations. In the following, however, we will see that in relativistic field theories the possible choice of commutations are restricted according to spin value of a particle field by the requirement of local causality. In [244], the proof was on more general ground using only the transformation properties of the field without assuming any quantization method. Because of the generality, the proof constrained only on integer spin explicitly. In this proof, let me introduce another more intuitive way of [246] based on the quantization by creation-annihilation operators instead. The
accent "ˆ" to indicate c number is presumed everywhere in the following.

In the footage of the quantization, the one-particle state $|m^2, p : s, \sigma\rangle$ is build up by a creation operator $a^\dagger$ as $a^\dagger(p, \sigma)|0\rangle$ which satisfy either of the statistics, i.e.

$$[a(p, \sigma), a^\dagger(p', \sigma')]_\pm = \delta_{\sigma\sigma'}\delta^3(p - p').$$  \hfill (A.41)

As it is expected from the discussion in the previous section, the transformation property of the one-particle state under Lorentz transformation, $U[\Lambda]$, becomes

$$U[\Lambda]|p, \sigma\rangle = N(\Lambda, p) \sum_{\sigma'} D^s_{\sigma\sigma'}[W]|p, \sigma'\rangle,$$  \hfill (A.42)

where $N(\Lambda, p)$ is a normalization factor to be determined by the conventional normalization $(p, \sigma|p', \sigma') = \delta_{\sigma\sigma'}\delta^3(p - p')$ and $D^s_{\sigma\sigma'}[W]$ is the representation matrix of little group given from that of $W^\mu$ generator defined in the previous section corresponding the spin $s$ state. This would be natural when we remind that $W^\mu$ prescribes the spin. In massive case, the matrix $D$ deduces to the pure rotation at the rest frame (called Wigner rotation), so that, as given in the previous section, it is nothing but $2s + 1$ dimensional representation of the rotation group. (In massless case it becomes that of $E_2$.) For $a^\dagger$ this relation leads

$$U[\Lambda]a^\dagger(p, \sigma)U^{-1}[\Lambda] = \left(\frac{(\Lambda p)^0}{p^0}\right)^{1/2} \sum_{\sigma'} D^s_{\sigma\sigma'}[W]a^\dagger(\Lambda p, \sigma').$$  \hfill (A.43)

Taking the Hermitian conjugate, we can get for annihilation operator $a$

$$U[\Lambda]a(p, \sigma)U^{-1}[\Lambda] = \left(\frac{(\Lambda p)^0}{p^0}\right)^{1/2} \sum_{\sigma'} D^s_{\sigma\sigma'}[-W]a(\Lambda p, \sigma'),$$  \hfill (A.44)

where $D[-W]$ denotes $D^{-1}[W]$. If necessary, we can also introduce antiparticle state with $b$ and $b^\dagger$, which transform as $a$ and $a^\dagger$.

Now we try to use them to construct a field, $\varphi(x)$, expressing a (asymptotic) free particle with one spin value $s$ and considered to be a Fourier transform of linear combinations of the creation and annihilation operators, eq. A.45. The complexity of the construction enters when we impose the requirement of its proper transformation under Lorentz transformation on the field, eq. A.46. (Remember that in general the representations of Lorentz
group have several spin values.)

\[
\varphi_i(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \sum_{\sigma'} \left( u_i(p, \sigma') a(p, \sigma') e^{ipx} + v_i(p, \sigma') b^\dagger(p, \sigma') e^{-ipx} \right)
\]

(A.45)

\[
U[\Lambda] \varphi_i(x) U^{-1}[\Lambda] = \sum_j D_{ij}^{(A,B)}[\Lambda^{-1}] \varphi_j(\Lambda x),
\]

(A.46)

where \(D[\Lambda]\) is some representation of Lorentz transformation, \((A, B)\), which includes spin \(s\) state in it. The question is now if we can find proper \(u\) and \(v\) so that \(\varphi\) satisfies eq. A.46. In the following, we are going to restrict possible choices of \((A, B)\) for \(s\) down only to \((s, 0)\). This can be justified by that fact any other \((A, B)\) state of spin \(s\) can be described by the appropriate derivation of \((s, 0)\) state so physically identical to \((s, 0)\). (The simplest example is vector field \((1/2, 1/2)\) with spin 0, of which behavior can be expressed by \(\partial_{\mu} \phi(x)\) with scalar field \(\phi\) of \((0, 0)\).) This simplification has also the advantage that \((s, 0)\) has only one spin state in it and the consideration to find \(u\) and \(v\) becomes mush simpler because we don’t need to be bothered by projections to the required \(s\) state. The general derivation of \(u\) and \(v\) for arbitrary \((A, B)\) is given in \([92]\). Therefore the index \(i\) and \(j\) in eqs. A.45 and A.46 are replaced by \(\sigma\) and \(\sigma'\).

As expressed in eqs. A.43 and A.44, the behavior of \(a\) and \(b^\dagger\) under Lorentz transformation depends on the individual momentum \(p\) so that the ordinary Fourier transform does not have a covariant nature. In order to achieve this, we have to look closer at the Lorentz transform behavior of the rotation matrix \(D[W]\), which depends on the momentum of the state as explained in the previous section. In case of \((s, 0)\), this can be cleared by the decomposition of \(D[W]\) into \(p\) dependent part and independent piece. Although I do not go into its detail, \(D[W]\) and \(D[-W]\) in eqs. A.43 and A.44 can be actually decomposed into three parts;

\[
D[W] = D^{-1}[L(\Lambda p)]D[\Lambda]D[L(p)],
\]

(A.47)

\[
D[-W] = D^{-1}[L(p)]D[\Lambda^{-1}]D[L(\Lambda p)],
\]

(A.48)

where all of the three are \((s, 0)\) \((2s + 1\) dimensional) expressions of the some Lorentz transform operators in the bracket \([\ ]\) and \(L(p)\) is a boost operator from its rest frame to \(p^\mu\). (More generally \(L(p)\) is defined as a boost to \(p^\mu\) from the ”standard” momentum \(k^\mu\), which appeared in the classification of little group, eq. A.18.) To be explicit, remembering the discussion in the previous section, \(D[L(p)]\) can
be expressed in $(s, 0)$ case as

$$D_{s\sigma}^s[L(p)] = \exp(-\hat{p} \cdot J^s \theta), \quad (A.49)$$

where $\hat{p}$ is the unit vector of $p$ direction, $\theta$ shows the rapidity ($\sinh \theta = |p|/m$) and $J^s$ is the $2s+1$ dimensional representation of the ordinal rotation.

After the decomposition, we can see from eqs. $A.43$ and $A.44$ that

$$U[\Lambda] \alpha(p, \sigma) U^{-1}[\Lambda] = \sum_j D^s_{\sigma,\sigma'}[\Lambda^{-1}] \alpha(\Lambda p, \sigma'), \quad (A.50)$$

$$U[\Lambda] \beta(p, \sigma) U^{-1}[\Lambda] = \sum_j D^s_{\sigma,\sigma'}[\Lambda^{-1}] \beta(\Lambda p, \sigma') \quad (A.51)$$

when we define

$$\alpha(p, \sigma) \equiv |2p^0|^{1/2} \sum_{\sigma'} D^s_{\sigma,\sigma'}[L(p)] a(p, \sigma'), \quad (A.52)$$

$$\beta(p, \sigma) \equiv |2p^0|^{1/2} \sum_{\sigma'} (D^s[L(p)]C^{-1})_{\sigma\sigma'} b^{\dagger}(p, \sigma'), \quad (A.53)$$

where $C$ is a constant unitary matrix.

With the linear combinations of those $\alpha$ and $\beta$, we can construct the required field $\varphi(x)$ which transforms properly under Lorentz transformation as

$$\varphi_{\sigma}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left( \xi \alpha(p, \sigma) e^{ipx} + \eta b^{\dagger}(p, \sigma) e^{-ipx} \right) \quad (A.54)$$

$$U[\Lambda] \varphi_{\sigma}(x) U^{-1}[\Lambda] = \sum_j D^s_{\sigma,\sigma'}[\Lambda^{-1}] \varphi_{\sigma'}(\Lambda x), \quad (A.55)$$

where $\xi$ and $\eta$ are to be fixed in the following. In terms of the original creation and annihilation operators, the field becomes

$$\varphi_{\sigma}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \sum_{\sigma'} \left( \xi D^s_{\sigma,\sigma'}[L(p)] a(p, \sigma') e^{ipx} + \eta (D^s[L(p)]C^{-1})_{\sigma\sigma'} b^{\dagger}(p, \sigma') e^{-ipx} \right) \quad (A.56)$$

It should be noted here that we did not assume any field equations except obvious Klein-Gordon equation $(\square - m^2) \varphi = 0$. Any other free field equations on $\varphi$ are nothing but an invariant record of which components are
superfluous. If the field has no anti-particle, simply set $\eta = 0$

Now we are ready to examine if the field $A.56$ satisfy local causality. The requirement of the local causality in a field theory is expressed with any asymptotic free field $\varphi(x)$ describing a particle as

$$\left[\varphi_i(x), \varphi_j(y)\right]_\pm = 0, \quad \left[\varphi_i(x) , \varphi_j(y)^\dagger\right]_\pm = 0, \quad \text{for } (x-y)^2 < 0,$$

(A.57)

where $[ , ]_\pm$ may be either the commutator or anti-commutator. This requirement comes from the fundamental necessity that all physical (observable) local quantities given from $\varphi(x)$, like gauge invariant or BRS singlet quantity, at space-like finite distance are commute. For a theory in which perturbative treatment is applicable, the locality requirement has deeper origin as Lorentz invariance of S matrix.

When we remind that in the particle interpretation with the language of creation and annihilation operators, they must satisfy

$$\left[a(p, \sigma), a^\dagger(p', \sigma')\right]_\pm = \delta_{\sigma\sigma'}\delta^3(p - p'), \quad \text{(A.58)}$$

$$\left[b(p, \sigma), b^\dagger(p', \sigma')\right]_\pm = \delta_{\sigma\sigma'}\delta^3(p - p'), \quad \text{(A.59)}$$

with all others vanishing. If we work out with these relations and eq. A.56 for the second of eq. A.57 (the first would be obvious), we can get

$$\left[\varphi_{\sigma}(x), \varphi_{\sigma'}(y)^\dagger\right]_\pm =$$

$$\frac{m^{-2s}}{(2\pi)^3} \int \frac{d^3p}{2p^0} \Pi^{s}_{\sigma\sigma'}(p) \times$$

$$\left( |\xi|^2 \exp(ip(x-y)) \pm |\eta|^2 \exp(-ip(x-y)) \right), \quad \text{(A.60)}$$

where the $(2s + 1) \times (2s + 1)$ matrix $\Pi^s(p)$ is given by

$$m^{-2s}\Pi^s(p) = D^s[L(p)]D^s[L(p)]^\dagger = \exp(-2\hat{p} \cdot J^s \theta') \quad \text{(A.61)}$$

with $\theta'$ defined by $\cosh \theta' = p^0/m$. The explicit expression of the matrix can be found in [246]. The important point is that it is written as polynomial function of $p$ and in fact it can be generally expressed as

$$\Pi^s(p)_{\sigma\sigma'} = (-1)^{2s} t^{\mu_1, \ldots, \mu_{2s}}_{\sigma\sigma'} p_{\mu_1} \cdots p_{\mu_{2s}}, \quad \text{(A.62)}$$

where $t$ is a constant symmetric traceless tensor. With this notation, eq.
A.60 can be rewritten as

\[ [\varphi_\sigma(x), \varphi_\sigma^\prime(y)]_\pm = \frac{1}{(2\pi)^{-3}}(-im)^{-2s} t_{\sigma_1\ldots\sigma_2}^\mu \partial_{\mu_1} \ldots \partial_{\mu_2} \times \]

\[ \int \frac{d^3 p}{2p^0} \left( |\xi|^2 \exp(ip(x-y)) \pm (-)^{2s}|\eta|^2 \exp(-ip(x-y)) \right) . \]  

(A.63)

The function

\[ \int \frac{d^3 p}{2p^0} \left( |\xi|^2 \exp(ip(x-y)) \pm (-)^{2s}|\eta|^2 \exp(-ip(x-y)) \right) \]  

(A.64)

is well known that it will vanish outside the light-cone if, and only if, the coefficients of \( \exp(ip(x-y)) \) and \( \exp(-ip(x-y)) \) are equal and opposite, i.e.,

\[ |\xi|^2 = \mp(-)^{2s}|\eta|^2 . \]  

(A.65)

This can be explicitly checked by direct calculations of the integration for both cases of \( \exp(ipx) \pm \exp(-ipx) \). In case of \( + \) sign, we can get

\[ \int \frac{d^3 p}{(2\pi)^3 2p^0} \left( e^{ipx} + e^{-ipx} \right) \]

\[ = \Delta(x) \quad \text{(invariant delta function)} \]

\[ = -\frac{1}{4\pi r} \frac{\partial}{\partial r} F(x^0, r) , \]  

(A.66)

where \( r = |x| \) and \( F(x^0, r) \) is

\[ F(x^0, r) = J_0(m\sqrt{x^2}) : x_0 > r \]

\[ = 0 : r > x_0 > -r \]

\[ = -J_0(m\sqrt{x^2}) : -r > x_0 , \]

where \( J_0 \) is the zeroth order of Bessel function of the first kind. This is pictorially shown in fig. A.1. Whereas in \( - \) case,

\[ \int \frac{d^3 p}{(2\pi)^3 2p^0} \left( e^{ipx} - e^{-ipx} \right) \]

\[ = \Delta^+(x) \]

\[ = \frac{1}{4\pi r} \frac{\partial}{\partial r} F_1(x^0, r) , \]  

(A.67)

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where $F_1(x^0, r)$ is

$$
F_1(x^0, r) = N_0(m\sqrt{x^2}) : x_0 > r
= -i H^1_0 : r > x_0 > -r
= N_0(m\sqrt{x^2}) : -r > x_0,
$$

where $N_0$ is the zeroth order of Bessel function of the second kind and $H^1_0$ is the zeroth order of the first Hankel function.

After all from eq. (A.65) we obtain the following two conclusions.

a) **Statistics:**
The equation makes sense only if

$$\mp(-1)^{2s} = 1. \quad (A.68)$$

Thus a particle with integer spin must be a boson, with a $-$ sign in eqs. (A.58) and (A.59), while a particle with half-integer spin must be a fermion with a $+$ sign in the eqs.

b) **Crossing:**
The equation requires also

$$|\xi| = |\eta|. \quad (A.69)$$

Thus every particle must have an antiparticle which enters into interactions with equal coupling strength.

### A.2.3 massless case and related subjects

Here in this subsection I’m going to give short remarks on the massless case, which was not treated in the above proof. The difficulty of the massless case comes from the different structure of the little group, as already suggested it becomes two dimensional Euclidean group $E_2$. Actually with the “standard” momentum $k^\mu = (\kappa, 0, \kappa)$, we can define the representation of the little group and the transformation properties exactly like eqs. (A.42), (A.43) and (A.44) also for massless state of $|0, p, s, \sigma\rangle$. (Note that $\sigma$ takes only either $s$ or $-s$.)

However, the construction of a covariant field with those creation annihilation operators like eq. (A.45) and (A.46) is not straightforward like massive case. As well discussed in [248], it is impossible to construct the covariant field which transforms properly according to a representation which includes
the spin $s$ under Lorentz transformation unless $\sigma$ or equally $s$ is equal to $-A + B$. In other cases, the coefficients $v_i$ and $u_i$ cannot be given to be consistent with eq. A.46. For example, we can construct $s = 0$ with $(0,0)$ as massless scalar field, $s = 1/2$ with $(1/2,0)$ as Wyle field, $s = 1$ with $(1,0) \oplus (0,1)$ as anti-symmetric tensor, and so forth. While we note that we can not compose the massless fields for the most important cases of

- a) $s = 1$ with $(1/2, 1/2)$ : massless vector field $A_\mu$
- b) $s = 3/2$ with $(1, 1/2)$ : massless Rarita-Schwinger field $\psi^\mu$
- c) $s = 2$ with $(1, 1)$ : massless symmetry tensor field $g_{\mu\nu}$.

These are the only fields which can mediates the interaction declining gently by inverse-square of the distance, called gauge fields. However, as well stated in [92, 92], there is one exception to introduce these fields to a theory. If the fields couples (only) to the conserved currents $X$, $\partial_\mu X^\mu = 0$, of the theory, these fields still can take part in the theory. This is because we can compensate the anomalous transformation behavior of those fields under Lorentz formation by the distinctive property of the coupling of the fields to the conserved currents. For the fields listed above, the coupling should be

$$A_\mu J^\mu, \quad \bar{S}_\mu \psi^{\mu}, \quad g_{\mu\nu} T^{\mu\nu} \quad \text{(A.70)}$$

because of their indices. Here the conserved quantities are the following. $J^\mu$ is the charge current related to (global) gauge symmetry, $S^\mu$ is the spinor current, related to supersymmetry, and $T^{\mu\nu}$ is the energy-momentum tensor,
related to translation symmetry. The couplings are nothing but the gauge couplings.

To see the anomaly compensation, take the simplest case of the vector field $A_\mu$. Without going into its detail, let us show the anomaly behavior of $u$ in eq. A.45 under Lorentz transformation. In $A_\mu$ case $u_\mu$ is expressed using the polarization vector $e_\mu(p, \sigma)$ as

\[ u_\mu(p, \sigma) \equiv (2p^0)^{-1/2} e_\mu(p, \sigma) , \quad (A.71) \]

\[ e^\mu(p, \sigma) = L^\mu_\nu(p)e^\nu(k, \sigma) \quad (A.72) \]

where $L(p)$ is a boost from the standard momentum $\kappa^\mu = (\kappa, 0, \kappa)$ of the massless case to $p$. To have a field $A^\mu$ properly transform under Lorentz transformation, the polarization vector $e^\mu$ should transform

\[ e^\mu(k, \pm 1)\exp(\pm i\theta) = D^\mu_\nu[W] e^\mu(k, \pm 1) , \quad (A.73) \]

where $\theta$ is the rapidity of the boost from $k^\mu$ to $p^\mu$ and $D[W]$ is again the representation of little group which, however, has the structure of $E_2$ in massless case. Using the property of $E_2$ structure, the possible choices of $e^\mu(k, \sigma)$ reduces to well-known form of

\[ e^\mu(k, \pm) = (0, 1, \pm i, 0)/\sqrt{2} . \quad (A.74) \]

While, again from the requirement of $E_2$, its transformations must become as

\[ D^\mu_\nu[W]e^\mu(k, \pm 1) = \exp(\pm i\theta) \left( e^\mu(k, \pm 1) + \frac{\alpha \pm i\beta}{\sqrt{k}} k^\mu \right) , \quad (A.75) \]

with $\alpha$ and $\beta$ arbitrary constants. Compared with eq. A.73 the second term in the bracket on the left side serves as anomalistic behavior of $e^\mu$ under Lorentz transform. Using eq. A.72 eq. A.75 can be rewritten as

\[ e^\mu(\Lambda p, \pm 1)\exp(\pm i\theta) = D^\mu_\nu[\Lambda] e^\nu(p, \pm 1) + p^\nu \Omega , \quad (A.76) \]

where $\Omega$ is a definite function of $p$ and $\Lambda$. Therefore the vector field $A^\mu$, constructed from the $e^\mu(p)$ by the Fourier transform of linear combinations of them, will transform as

\[ U[\Lambda] A^\mu(x) U^{-1}[\Lambda] = \Lambda^\nu_\mu A^\nu(\Lambda x) + \partial^\mu \Omega(x, \Lambda) . \quad (A.77) \]

Indeed $A^\mu$ behaves non-covariant way, if $A^\mu$ is only coupled with the conserved current $J^\mu$, $\partial_\mu J^\mu = 0$, in the Lagrangean, i.e. $A_\mu J^\mu$, this anomalistic
term does not play effective role in its dynamics thanks to the total differentiation in the term. Besides, the possibility of the appearance of spin 0 field of \((1/2, 1/2)\) is also wiped out by this coupling. \((A^\mu \) cannot be spin 0 field of \((1/2, 1/2)\)). With the help of this compensation with gauge coupling, \(A^\mu\) can be implemented in the theory as spin 1 field without anomaly behavior. Moreover, as a result of arbitrariness of the choice of gauge, eqs. \(\text{[A.75]}\) says there is a freedom in the definition of polarization vector by the additional term proportional to the “standard” null vector of the polarization. As we see in chapter \(\text{[B]}\) this fact is greatly used to construct the polarization vectors of a gauge, axial (physical unitary) gauge.

Additionally there is a limitation on the possible candidates of the conserved currents in relativistic field theory. Coleman and Mandula \(\text{[251]}\) and Haag, Lopuszanski and Sohnius \(\text{[252]}\) exhausted all the possible symmetry types of the S-matrix in the positive metric (physical) space based on the greatly general assumptions which the theory should have. Following their theorems, possible non-scalar charges are only Poincare generators \(P^\mu\) and \(J^{\mu\nu}\) and spinor (supersymmetry) charges \(Q^{i\alpha}\) and scalar charges are only the generators of compact groups of the theory. The proofs of their theorems can be followed pedagogically in \(\text{[253]}\). Note that \(J^{\mu\nu}\) cannot be the source charge of any massless particle since it is not invariant under translations. Therefore, we can conclude that the possible candidates of massless gauge fields and its couplings are those in eq. \(\text{[A.70]}\). This leads also that if massless particles with spin \(s \geq 5/2\) exist, they can neither couple to other particles nor to themselves in the infrared (long distance) limit; i.e. the infrared particles must be free. For this statement we can refer to the chapters dealing with infrared properties of the gauge fields in \(\text{[92]}\).

As the final remark, because the discussions of spin-statistics theorem was based on the physical requirement of the locality of the observables, this relations can be violated for particles which can never appear in physical space. Actually the ghost scalar fields \(c^a\) or \(\bar{c}^b\), which play important role in non-Abelian theory like QCD are formulated with

\[
[c^a(p), \bar{c}^b(p')]_+ = \delta^{ab} \delta^3(p - p').
\]

For the properties of these particles, refer to \(\text{[254]}\) or \(\text{[10]}\).
Appendix B

Gluon Polarizations in Light-Cone Gauge

In this appendix, I’m going to show the properties of the key ingredient of the ladder expansion, light-cone gauge. The first section is devoted to show its peculiar nature of the absence of FP ghosts. In the second section, I’m going to derive the helicity property of the gluon in the light-cone gauge. Along with the result, I listed up the projection operators for unpolarized and polarized cases in the last section, which are needed for actual calculations of the ladder expansion.

B.1 Properties of Light-Cone Gauge

Light-cone gauge is the special case of the axial gauge expressed as $n \cdot A(p)$, where $n$ is a fixed vector, which appears in the path integral as the power of $-\frac{1}{2a}[n \cdot A]^2$ similarly to the usual covariant one, $-\frac{1}{2a}[\partial \cdot A]^2$. What we are now interested in is the simplest case, $\alpha = 0$ (Landau gauge), in which works the cancellation of FP ghosts in non-Abelian case [255]. The “axial” means roughly that we are not going to deal with polarizations which lie in the 4-plane of the reaction $p, n$. Since the emitted gluons in the average belong to just on this plane, it means that we are effectively throwing away the non-physical polarizations. To see this, let us write down the propagator of the gluon with a momentum $p$ in this gauge;

$$G^{\mu \nu}(p) = \frac{i}{p^2 + i\epsilon} d^{\mu \nu}, \quad d^{\mu \nu}(p) = -g^{\mu \nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{(p \cdot n)} - n^2 \frac{p^\mu p^\nu}{(n \cdot p)}.$$  \hspace{1cm} (B.1)

This propagator has the following properties:

$$d^{\mu \nu}(p) = 2 \cdot p_\mu d^{\mu \nu}(p) \rightarrow 0 \text{ on the mass shell} \hspace{1cm} (B.2)$$
This clearly shows the only two physical polarizations, $p_\mu e^\mu = 0$, propagate. (For comparison, in the Feynman gauge $d^{\mu\nu} = -g^{\mu\nu}$, we have $d^{\mu\nu} = 4 , p_\mu d^{\mu\nu} \neq 0$.)

In non-Abelian case, we have considerable advantage of having such an unitary gauge [82]. Because physical interactions are closed only in physical Hilbert space [10], many practical calculations of ref. [3, 27, 26, 256] was performed in this gauge relieved from the effect of the complex ghost mixture. The idea of the gauge is rather old and dates back to [257, 255]. As the general feature of the unitary gauges, there are several subtleties in actual calculations, like breaking of power counting or spurious divergences, depending on chosen gauge. All of these comes from the uncleanness of treatment of $1/(p \cdot n)$ factor appearing in gluon propagator eq. [3,1] Still, the gauge invariance of the physical S-matrix implies that these anomalous behaviors must cancel in the sum of all diagrams. Actually in [258], additional potential singularities of the axial gauge ($n^2 = 0$) are confirmed to be canceled and the S-matrix keeps gauge independent and for $n^2 = 0$ in [102, 103]. Overview of the subtleties of the axial type gauge can be found in [104].

Moreover, as indicated in chapter [2,3] this physical gauge plays crucial role in the discussion with the ladder type diagram, which deduces parton picture. In the category of the axial gauge, the light-cone gauge is defined with the additional condition $n^2 = 0$ set to $n$, which is implemented in [259] (for Abelian model). The light-cone gauge much loose the complexity of diagram calculation though there are still subtleties related to $(p \cdot n)^{-1}$ treatment. Additionally the helicity structure and the physical denotation becomes much evident, which is significant for the calculations of spin dependent quantities. In the next section, I’m going to look closer to the helicity structure.

### B.2 Polarization Vectors of Gluons

To obtain the polarization vectors of gluons in the light-cone gauge, here I would like to introduce spinor-helicity formalism, introduced in [260]. This formalism relates the polarization vector to massless spinors and gives clear descriptions of them. The essential keynote of this formation is that the physical Hilbert space of a massless vector is isomorphic to that of a massless spinor (up to a $Z_2$ transformation between integer spin and half-integer spin), because both lie in one-dimensional representations of $SO(2)$, the little group of $E_2$ as we saw in Appendix A.
Let $\psi(p)$ a massless Dirac spinor;

$$\not{p}\psi(p) = 0, \quad p^2 = 0.$$  \hfill (B.3)

We define the two helicity states of $\psi(p)$ by the two chiral projections;

$$\psi_{\pm}(p) = \frac{1}{2}(1 \pm \gamma_5)\psi(p) \quad \text{(B.4)}$$

Note that in case of massless, helicity eigen values are simply consistent to that of chirality. Then define $|p\pm\rangle$ and $\langle p\pm| \,$ and convolution scalars as

$$|p\pm\rangle \equiv \psi_{\pm}(p), \quad \langle p\pm| \equiv \overline{\psi_{\pm}}(p), \quad \text{(B.5)}$$

The normalizations are taken as;

$$\langle p\pm | \gamma^\mu | p\pm \rangle \equiv 2p^\mu. \quad \text{(B.7)}$$

From the properties of Dirac algebra, we can find the following relations;

$$\langle p + | p + \rangle = \langle p - | p - \rangle = \langle pp \rangle = [pp] = 0 \quad \text{(B.8)}$$
$$\langle pq \rangle = -\langle qp \rangle, \quad [pq] = -[qp] \quad \text{(B.9)}$$
$$\langle A + | \gamma_\mu | B + \rangle \langle C - | \gamma_\mu | D - \rangle = 2[AB]\langle CD \rangle \quad \text{(B.10)}$$
$$|p\pm\rangle \langle q \pm | = \frac{1}{4}(1 \pm \gamma_5)\gamma^\mu\langle q \pm | \gamma_\mu | p\pm \rangle \quad \text{(B.11)}$$
$$\langle pq \rangle^* = [qp]. \quad \text{(B.12)}$$

The isomorphism is realized through a linear transformation which relates helicity-like vectors and fermions:

$$\epsilon^{+}_\mu(p) = A\overline{\psi_{+}(p)}\gamma_\mu v, \quad \epsilon^{-}_\mu(p) = (\epsilon^{+}_\mu(p))^*, \quad \text{(B.13)}$$

where $\epsilon^\pm(p)$ is the polarization vectors of an positive-energy massless vector with momentum p, $v$ is an arbitrary Dirac spinor, and A is a normalization constant, needed to set the usual normalization conditions;

$$\epsilon^{+}(p) \cdot \epsilon^{+}(p) = 0, \quad \epsilon^{+}(p) \cdot \epsilon^{-}(p) = -1. \quad \text{(B.14)}$$

In this expression, the gauge invariance associated with the massless vector appears as the arbitrariness in the choice of the spinor $v$ (although this
procedure does not exhaust all the possible gauge. Let us choose a spinor $v(k)$ to satisfy the properties,

$$k v(k) = 0, \quad k^2 = 0, \quad k \cdot p \neq 0. \quad (B.15)$$

$k$ is usually referred as reference momentum. Then, after the proper normalization, $e^\mu$ can be expressed as

$$e^\mu_\pm (p, q) = e^{\pm \phi(p,q)} \frac{\langle p \pm | \gamma^\mu | q \pm \rangle}{\sqrt{2} \langle q \mp | p \pm \rangle}. \quad (B.16)$$

Then we can see

$$p_\mu e^\mu_\pm (p, q) = q_\mu e^\mu_\pm (p, q) = 0, \quad (B.17)$$

$$f_\pm (p, q) = \frac{\sqrt{2}}{\langle q \mp | p \pm \rangle} (| p \mp \rangle \langle q \mp | + | q \mp \rangle \langle p \pm |). \quad (B.18)$$

Especially, note that usual polarization $e^\mu_\pm (k)$ of the covariant gauge, like eq. (A.74) can be expressed as

$$f_\pm (k) = \left( | k \pm \rangle \langle k \mp | + | k \mp \rangle \langle k \mp | \right) \quad (B.19)$$

with $\tilde{k} = (k, -k)$. (If we put $k_\mu$ as the "standard" vector, we can exactly reconstruct eq. (A.74).) Then eq. (B.16) can be rewritten from (B.18) as

$$e^\mu_\pm (k, q) = e^{\pm \phi(k,q)} \left( e^\mu_\pm (k) + \frac{i}{\sqrt{2k} \langle q \mp | k \pm \rangle} k^\mu \right). \quad (B.20)$$

Compared with the right hand side of (A.75) and the following discussion of the gauge coupling, we can clearly see that the choice of polarizations is nothing but a fix of the freedom of the choices of $\alpha$ and $\beta$ in $E_2$, in other words, a choice of gauge.

Using eqs. (B.4) - (B.12) and after some trace algebra of $\gamma$ matrices, we can get

$$e^\mu_\pm (p, q) e^{\nu_\pm} (p, q) = \frac{1}{2} \left( -g^{\mu\nu} + \frac{p^\mu q^\nu + p^\nu q^\mu}{k \cdot q} \right) \pm i \epsilon^{\mu
\nu\sigma\rho} \frac{q_\rho p_\sigma}{2p \cdot q} \quad (B.21)$$

and also the completeness

$$\sum_{h = \pm} e^\mu_h (k, q) e^{\nu_h} (k, q) = -g^{\mu\nu} + \frac{p^\mu q^\nu + p^\nu q^\mu}{p \cdot q}. \quad (B.22)$$
From the discussion of the previous section, we see the polarization vectors eq. B.16 are equivalent to those of the light-cone gauge with a fixed null vector \( q \).

As additional remark, this choice of gauge or equivalently polarization vectors is also great advantage to simplify Feynman diagram calculations by virtue of eqs. B.17 (Note that the choice of null vector \( q \) was arbitrary), as we can see the concrete example in [142], and with the help of supersymmetry [201] it is extensively used for multi-parton jet calculations [262].

### B.2.1 Projection Operators in Ladder Expansion

As the results of the above discussion, we can get the following simple projection operators, which was introduced in section 2.3 for polarized (helicity) case and unpolarized case when we set \( q^\mu = n^\nu \), where \( n^\nu \) is the null vector which regulates the light-cone gauge.

In addition to the gluon polarization, it would be complementary to show the completeness of quark field in helicity basis manifestly,

\[
\begin{align*}
  \mathfrak{u}(p, \sigma)\bar{\mathfrak{u}}(p, \sigma) & = \mathfrak{v}(p, -\sigma)\bar{\mathfrak{v}}(p, -\sigma) = \frac{1}{2}(1 - \sigma \gamma_5) \slashed{p}. 
\end{align*}
\]  

(B.23)

After all, we can obtain the following projection operators onto physical states, for unpolarized case,

\[
\begin{align*}
  U_q = \mathfrak{U}_q &= \frac{1}{4n \cdot k} \gamma^\mu, & L_q = \mathfrak{L}_q &= \frac{1}{d-2} \left[ -g^{\mu\nu} + \frac{n^\nu p^\mu + n^\mu p^\nu}{p \cdot n} \right], 
\end{align*}
\]  

(B.24)

(B.25)

and for polarized (helicity) case,

\[
\begin{align*}
  \Delta U_q = -\Delta \mathfrak{U}_q &= \frac{1}{4n \cdot k} \gamma_5 \gamma^\mu, & \Delta L_q = -\Delta \mathfrak{L}_q &= \frac{1}{d-2} \left[ -g^{\mu\nu} + \frac{n^\nu p^\mu + n^\mu p^\nu}{p \cdot n} \right], 
\end{align*}
\]  

(B.26)

(B.27)

where the meaning of each parameter can be found in section 2.3.
Bibliography


[58] Yoshimitsu Imazui. QCD analysis of parton helicity distributions with HERMES SIDIS data.


